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EMBODIED COGNITION AS GROUNDING FOR SITUATEDNESS AND CONTEXT IN MATHEMATICS EDUCATION

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ABSTRACT: In this paper we analyze, from the perspective of "Embodied Cognition", why learning and cognition are situated and context-dependent. We argue that the nature of situated learning and cognition cannot be fully understood by focusing only on social, cultural and contextual factors. One must also take into account the non-arbitrary biological and experiential constrains that shape social activity and language, and through which cognition and learning are realized in a genuine embodied process. The bodily-grounded nature of cognition provides foundations for situatedness, entails a reconceptualization of cognition and mathematics itself, and has important consequences for mathematics education. After framing some theoretical notions of embodied cognition in the perspective of modern cognitive science, we analyze a case study continuity of functions. We use conceptual metaphor theory to show how embodied cognition, while providing grounding for situatedness, gives fruitful results in analyzing the cognitive difficulties underlying the understanding of continuity.

1. INTRODUCTION

An important goal of mathematics education is to understand the thinking involved in doing and learning mathematics. In recent years, it has become widely accepted that the learning and practice of mathematics are not purely intellectual activities, isolated from social, cultural, and contextual factors (Lave, 1988; Collins, Brown & Newman, 1989; Cobb, 1994; Confrey, 1995). Instead, it has been acknowledged that learning and teaching take place, and have always taken place, within embedding social contexts that do not just influence, but essentially determine the kinds of knowledge and practices that are constructed (Lave & Wenger, 1991; Rogoff, 1990; Walkerdine, 1982). Perspectives that focus on the social and contextual nature of knowledge, and that make the embedding situation prominent in the analysis of cognition, have been labeled as "situated". Research and theoretical frameworks based on a situated approach to cognition insist that linguistic, social, and interactional factors be included in any account of subject matter learning, including the learning of mathematics. The hallmark of this approach is that it "considers processes of interaction as basic and explains individual cognitions and other behaviors in terms of their contributions to interactive systems" (Greeno, 1997, p. 15).

These approaches have yielded many important results, and have helped to move the analysis of learning beyond a narrow focus on individual and "internal" cognitive processes. Yet we would argue that the nature of situated learning and cognition cannot be fully understood by attending only to contextual or social factors considered as inter-individual processes. Thinking and learning are also situated within biological and experiential contexts, contexts which have shaped, in a *non-arbitrary* way, our characteristic ways of making sense of the world. These characteristic ways of understanding, talking about, and acting in the world are shared by humans by virtue of being interacting members of the same species, co-existing within a given physical medium. The overall aim of this paper is to present elements of a theory which focuses on how human cognition is bodily-grounded, that is, *embodied* in living structures and processes that are immersed in shared biological and physical contexts, and to examine the ways in which this embodiment helps to determine the nature of mathematical understanding and thinking. In particular, we intend to investigate foundational aspects of situatedness by bringing in alternative approaches to orthodox cognitive science that focus on embodiment. One of our claims is that the situated cognition approach, as valuable as it is, leaves open important questions - what is the grounding for situated knowing and learning? What is the basis of social situatedness? We share with the situated learning approach the belief that knowledge and cognition exist and arise within specific social settings, but we go on to ask what it is that makes possible the mutual intelligibility underlying shared social understandings. Our claim is that the grounding for situatedness comes from the nature of shared human bodily experience and action, realized through basic embodied cognitive processes and conceptual systems.

When taken seriously, genuine embodiment entails a reconceptualization of the nature of cognition and of mathematics itself, with implications for teaching (Lakoff & Núñez, 1997). A first implication is that we must leave behind the myth of mind-free mathematics as being about eternal, timeless truths, a legacy of Plato and Descartes. From an embodied perspective, the notion of an objective mathematics, independent of human understanding no longer makes sense. Another implication is that we are required to give an account of a mind-based mathematics, including an explanation of its stability and efficacy, in terms of the human bodily-based and situated conceptual systems from which it arises. Such an account should be useful in understanding problems in the teaching and learning of mathematics, and in designing instruction supportive of an embodied and socially-situated mathematics.

In this paper, we will first frame the notion of embodied cognition within an intellectual and theoretical context, and elaborate the relationship between situated learning and embodied cognition. Next, we will present an example of difficulties in mathematics learning, focusing on the idea of continuity; this topic offers a case study that can be fruitfully understood from an embodied cognition perspective. We close with a discussion of implications and directions for further empirical and theoretical research which utilizes the work done in embodied cognition.

2. FRAMING EMBODIMENT THEORETICALLY AND HISTORICALLY

The subject of this special issue, known variously as "situated learning," "situated activity," or "situated cognition," has offered a productive approach to understanding learning and teaching, including the learning and teaching of mathematics. This perspective stresses the idea that learning always occurs within a specific context, and, most importantly, within social contexts in which meanings are negotiated and constructed in on-going interactions among the participants. Lave and Wenger make this claim explicit when they state that "there is no activity that is not situated," and when they note the perspective's "emphasis on comprehensive understanding involving the whole person rather than 'receiving' a body of factual knowledge about the world; on activity in and with the world; and on the view that agent, activity, and the world mutually constitute each other" (Lave & Wenger, 1991, p. 33).

The situated learning perspective was welcomed by educational researchers and theorists as a richer and more appropriate means of addressing cognition than that offered by the formal, cognitivist models of mainstream cognitive science in the 1970s. This latter approach, strongly influenced by the objectivistic tradition of analytical philosophy and by functionalism, focused on learning as a process of individual reasoning, often explained in computational terms. Researchers in mainstream cognitive science at that time held that in explaining human cognition, it was necessary (and sufficient) to "posit a level of analysis wholly separate from the biological or neurological, on the one hand, and the sociological or cultural, on the other " (Gardner, 1985, p. 6). This separate level of analysis focused on the individual as a processor of information, and characterized reasoning as the manipulation of arbitrary symbols. Under this view, symbols gain meaning from being associated with an objective reality, which is modeled in the mind by internal representations corresponding, to greater or lesser degrees of accuracy, with that external reality. This view of cognition became (and still is) pervasive in cognitive psychology (see for example, Howard, 1983; Sanford, 1985; Eysenk & Keane, 1992), and was subsequently adopted by researchers in mathematics education seeking a paradigm for understanding mathematical thought. An expression of this view in a contemporary account of mathematics learning is found, for example, in English & Halford (1995), who state, "Cognitive processes entail operations on mental representations, which are internal mental structures that correspond to the structure of a segment of the world" (ibid., p. 21).

Early mainstream cognitive science (cognitivism)

Early mainstream cognitive science cognitivism maintained the Cartesian dualism which holds that the mind is an abstract entity, separate from and transcending the body. Reasoning (including mathematical thought) is also non-corporeal, timeless, and universal. Concepts the products of reasoning are abstract, not limited by physical or bodily realities, and they allow us to understand the structure of the supposedly real external world. Cognitivism is thus based on objectivism, the doctrine that assumes transcendental ontological truths that are independent of human understanding (Núñez, 1995). Under this view of knowledge, an objectivist would hold that the Pythagorean theorem, for example, "is" true and valid in this and in any universe, irrespective of the existence of human beings.

This framing of mind and reasoning is consistent with a model of cognition as computation, and with a functionalist stance which intends to study the mind in terms of the mental functions it performs without seriously considering how the brain and body actually work. But deeper than that, functionalism postulates that in order to explain cognition, one merely needs to specify a mechanism that performs the functions associated with a given cognitive activity. Functionalist scholars claim that, in theory, nothing would prevent such a mechanism from being instantiated in a computer program; thus, an entailment of this perspective is the belief that genuine cognition can be engineered into non-living computer systems (e.g., so-called expert systems in Artificial Intelligence). As Gardner states, "Not only are computers indispensable for carrying out studies of various sorts, but, more crucially, the computer also serves as the most viable model of how the human mind functions" (Gardner, 1985, p. 6). Herbert Simon, often identified as a founder of cognitive science, defines the discipline as "the study of intelligence and intelligent systems, with particular reference to intelligent behavior as computation" (Simon & Kaplan, 1989, p. 1). Although this strong view is not as popular as it once was, it has left a tremendous amount of residual objectivism and computer functionalism in the more subtle approaches of current mainstream cognitive science, such as PDP connectionism and artificial neural networks approaches.

The limitations, both theoretical and empirical, of cognitivism have become apparent in the 25 years since it became prominent. For example, it has been unable to satisfactorily model or account for everyday cognitive phenomena such as common sense, sense of humor, and natural language understanding. In addition, the information-processing models that came out of early mainstream cognitive science bore little resemblance to the observed processes of real life problem-solving and learning found either inside or outside the classroom (Rogoff & Lave, 1984; Lave, 1988; Confrey, 1990; Nunes & Bryant, 1996). Furthermore, the objectivism of mainstream cognitive science was incompatible with premises of radical constructivism, which does not assume a pre-determined reality that is straightforwardly accessed by the observer or learner (von Glasersfeld, 1990; Cobb, Yackel & Wood, 1992; Cobb, 1994). As an unfortunate result, researchers in mathematics education concerned with developmental, social, and cultural factors have in general rejected cognitive science as a whole, assuming that it had little to offer. If we consider only early mainstream cognitive science, this may be true. However, alternative approaches to the scientific study of human mind have emerged within cognitive science itself, approaches based on different assumptions about the nature of thought, knowledge, and human activity in the world. In our view, certain of these alternative approaches have much to offer to mathematics education.

Alternative approaches within cognitive science: Embodiment

Motivated by efforts to overcome the theoretical and empirical limitations of cognitivism, a number of alternatives emerged in the 1980s. These alternative approaches are based on work in a number of different disciplines, and do not share precisely the same theoretical commitments or research foci. However, they do have in common a number of core features that distinguish them from early mainstream cognitive science. These features include a commitment to investigating cognition as the product of on-going action on the part of an organism (or person) within an environment, rather than as a disembodied abstract process, and thus, they feature a specific focus on physical embodiment. In contrast to the classic computational model of early mainstream cognitive science, which is based implicitly on the serial computer with its centralized "executive", the newer paradigms stress the decentralization of cognitive processes and their realization through massively-parallel, ongoing simultaneous processes, a model inspired by neurological functioning. Finally, these approaches all emphasize the interaction of the multiple components of the physical implementation, again, conceiving cognition and activity as embodied processes.

But here one needs to be careful, since ideas related to the term "embodiment" are used in a number of different ways in the study of the mind and cognitive science, and these varied uses at times reflect fundamental theoretical differences. For some, they are about the phenomenological aspects of the human bodily experience (Merleau-Ponty, 1945; diSessa, 1983), and the resulting psychological manifestations (Rosch, 1994). Others focus on the organization of bodily action under principles of non-linear dynamics (Thelen & Smith, 1994). Certain theorists stress the unconscious aspects of bodily experience that underlie cognitive activity and linguistic expression (Johnson, 1987; Lakoff, 1987) and others bring in embodiment as a crucial paradigm in anthropology (Csordas, 1994: Lock, 1993). Yet others emphasize the biologico-structural codefinition that exist between organisms and the medium in which they exist, from which cognition results as an enactive process (Maturana and Varela, 1987). Along these lines, some stress the importance of the supra-individual biological processes that underlie high level cognition (Núñez, 1997). Notions of embodiment are even explicitly used in the design of responsive and adaptive non-living systems, where what is essential are the functional aspects of sensory-motor primitives, as built into integrated robotic mechanisms, controlled in real time by parallel computers (Brooks & Stein, 1993). Embodiment is also featured explicitly in computer modeling through the implementation of neurally-plausible structured connectionist computer models of cognitive linguistic activity (Feldman et al., 1996; Regier, 1996). Although all these approaches share the core features of embodiment identified in the previous paragraph, each highlights different characteristics of what is called "embodiment," and many differ in terms of important theoretical and philosophical implications.

In the current context, we are especially interested in those alternative approaches that endorse a notion of embodiment that challenges the objectivistic, dualist, and functionalistic traditions of mainstream cognitive science. In particular, our analysis builds on the work by Rosch in cognitive psychology (Rosch, 1973, 1994); Varela, Thompson & Rosch, 1991); Edelman (1992) in neuroscience; Maturana and Varela in theoretical biology (Maturana & Varela, 1987); Winograd & Flores (1986) in computer science; and more explicitly on the work by Lakoff and Johnson in cognitive linguistics (Lakoff & Johnson, 1980; Johnson, 1987; Lakoff, 1987). All these scholars share a focus on the intimate relation between cognition, mind, and living bodily experience in the world. They are all concerned with the embodied nature of thought, and in doing so, they reject dualism, objectivism, and functionalism: Dualism, because mind and body are not seen as separable; objectivism, because truths are not seen as mind-independent or transcendent; and functionalism, because they reject the claim that the specification of a mechanism that performs a function fully accounts for a given cognitive process. Instead, the specific situation (cultural, social, biological, historical) within which the cognitive activity occurs represents an inextricable element of any explanation of that activity.

Within these alternative paradigms in cognitive science, the knower and the known are codetermined, as are the learner and what is learned. Thus, cognition is about enacting or bringing forth adaptive and effective behavior, not about acquiring information or representing objects of an external world. The potential of this perspective for building a more satisfactory account of the human thinking is expressed by Varela, Thompson and Rosch, when they state, "If we wish to recover common sense, then we must invert the representationist attitude [of a pre-given world] by treating context-dependent know-how not as a residual artifact that can progressively eliminated by the discovery of more sophisticated rules, but as, in fact, the very essence of creative cognition" (Varela, Thompson & Rosch, 1991, p. 148). Here cognition is viewed as a living phenomenon, realized via a process of codetermination between the organism and the medium in which it exists. Rather than positing a passive observer taking in a pre-determined reality, these paradigms hold that reality is constructed by the observer, based on nonarbitrary culturally determined categories as well as individual bodily experience. This experience results in cognitive processes and concepts which are not abstract or transcendent, but instead fully embodied. Lakoff describes such concepts: "A concept is embodied when its content or other properties are motivated by bodily or social experience... Embodiment thus provides a nonarbitrary link between cognition and experience" (Lakoff, 1987, p. 154).

3. GROUNDING SITUATEDNESS IN EMBODIMENT

After this brief description of the concept of embodiment and its theoretical foundations, we can see that the idea of bringing bodily experience into the account of cognition and learning plays an important role in many sub-disciplines within current cognitive science. At this point, we would like to clarify our perspective on what we mean, and do not mean, when we use the term embodiment. From our perspective, embodiment is not simply about an individual's conscious experience of some bodily aspects of being or acting in the world (e.g., the first time we went skating). Nor does it refer just to explicit physical manipulation of specific concrete objects (e.g., playing with Cuisinaire rods or pattern blocks). Although there is a relationship between such experiences and the technical concept of embodiment, an embodied perspective does not simply offer a prescription for ways to teach a certain subject matter in a "concrete" way, for example, by encouraging the increased use of manipulatives in the teaching of mathematics. Although embodiment may provide a theoretical grounding for understanding the teaching and learning of "realistic mathematics", it is not directly concerned with realistic or "contextualized" subject matter teaching. Rather embodiment provides a deep understanding of what human ideas *are*, and how they are organized in vast (mostly unconscious) conceptual systems grounded in the peculiarities of the living brains and bodies of the animals we are.

Johnson (1987) gives a nice example of how basic and universal bodily experience serves as the grounding for abstract understanding in his discussion of the experience of balance. The experience of balance is part of our everyday life and makes possible our experience of the world as well as our survival in it. The ongoing experience of balancing our bodies (and our physical selves), is so basic and pervasive that we are rarely aware of it. Balancing is an activity we learn with our bodies from very early ages by acting, existing and developing in the world. It is not learned by acquiring abstract rules or algorithms. Moreover, the experience of balance is so basic that all homo sapiens no matter when and where they live on earth have had it. As such, it is one of a class of deep, unconscious, yet pervasive bodily-based experiences providing a space of commonalties that makes up the ground for shared human sense-making.

We build up the meaning of balance through the active ongoing experience of bodily equilibrium and loss of equilibrium. Along with this process, we start making sense of related systemic bodily experiences; for example, the feeling that our fingers are not warm *enough*, our the mouth is *too* dry, and so on. In these moments, our bodies naturally try to compensate by inducing our fingers to move and raise their temperature, or by secreting saliva (and eventually in orienting our bodily activity to look for water). Our understandings of "too much", "not enough", or "out of balance" are pre-conceptual, non-formal, and non-propositional. Sense-making is built up before we actually have any formal or abstract concepts of quantity or "balance". The meaning of balance begins to emerge through our acts of balancing and through our experience of systemic processes and states within our bodies in the world. This meaning is intimately related to these experiences, and in particular, to the *image-schematic* structures that make those experiences coherent and significant for us. As we will see, the image schematic structure of these pre-conceptual forms of sense-making are often cognitively extended via conceptual mappings (conceptual metaphors, conceptual metonymies, conceptual blends, etc.) to create new shared meanings. These (unconscious) mappings preserve the original image-schematic structure in various ways to give raise to more abstract meanings for balance, such as those underlying concepts like "balancing" colors" in a picture, "balancing" a checking account, or "balancing" a system of simultaneous equations. Indeed, it is our thesis that the basis of a great deal of our mathematical knowledge lies is in such conceptual mappings; as Lakoff & Núñez state, "much of what is 'abstract' in mathematics...concerns coordination of meanings and sense making based on common image-schemata and forms of metaphorical thought. Abstract reasoning and cognition are thus genuine embodied processes" (Lakoff & Núñez, 1997, p. 30).

Some technical concepts

Theories of embodiment have generated an important number of technical terms. In the field of cognitive semantics alone, terms that characterize basic cognitive processes such as image-schemata, conceptual metaphor, prototype systems, radial categories, frame semantics, basic-level concepts, conceptual blends, and many others, are essential for study within the field. Because of space constraints, we will focus only on the first two: Image schemata and conceptual metaphor.

Image-schemata. Image schemata are basic dynamic topological and orientation structures that characterize spatial inferences and are mostly used in the understanding of spatial relation concepts. They are used in complex spatial relations concepts, and are projected by conceptual mappings (metaphors) onto concepts in abstract domains. It is important to mention that image-schemata are not static objectivistic propositions that characterize abstract relations between symbols and objective reality (like those specified by formal language). Rather they are dynamic recurrent regular patterns of ongoing ordering actions, perceptions, and conceptions. These patterns emerge as meaningful structures for us mainly through the bodily experience of movement in space, manipulation of objects, and perceptual interactions.

Image schemata are structures of an activity by which we organize our experience in ways that we can comprehend. "They are a primary means by which we construct or constitute order and not mere passive receptacles into which experience is poured" (Johnson, 1987, p. 29). Some examples are the container schema (underlying concepts like IN and OUT); source-path-goal schema (TO and FROM); contact schema; and verticality schema. Many basic concepts are built on combinations of these schemata. The concept ON, for example, uses three basic schemata: verticality, contact, and support. As we will see later, by preserving the original inferential structure, image schemata are extended through specific cognitive mechanisms to make abstract concepts possible. For instance, within mathematics, Boolean logic is an extension of the container schemata, realized through a conceptual metaphorical projection of the logic of containers. This metaphorical projection preserves the original inferential structure of IN, OUT, and transitivity, developed origninally via physical experiences with actual containers, and later unconsciously mapped to a set of abstract mathematical concepts.

Conceptual metaphor. Conceptual metaphors are cross-domain "mappings" that project the inferential structure of a *source domain* onto a *target domain*. Such "projections" or "mappings" are not arbitrary and can be studied empirically, and stated precisely. They are not arbitrary, because they are motivated by our everyday experience - especially bodily experience. Research in contemporary conceptual metaphor theory has shown that there is an extensive conventional system of conceptual metaphors in every human conceptual system. These theoretical claims are based on empirical evidence from a variety of sources, including psycholinguistic experiments, generalizations over inference patterns, the study of historical semantic change, of spontaneous gestures, and of American sign language (Lakoff, 1993).

It has been found that metaphorical mappings are not isolated, but occur in complex systems and combine in complex ways. As with the rest of our conceptual system, our system of conventional conceptual metaphors is effortless and lies below the level of conscious awareness (when we consciously produce novel metaphors, we utilize the mechanisms of our unconscious conventional metaphor system). Unlike traditional studies of metaphor, contemporary embodied views don't see conceptual metaphors as residing in words, but in thought. Metaphorical linguistic expressions thus are only surface manifestations of metaphorical thought. What is very important for the study of abstract conceptual systems, such as mathematics cognition and learning, is that the inferential structure of the source domain is preserved in each mapping onto a target domain. That is, the image-schema structure of the source domain is preserved in the mappings.

Other than conceptual metaphors, also conceptual metonymies, or "metonymic mappings" have been studied. Such mappings link two elements in a single conceptual schema, in such a way that the first *stands for* the second. An everyday linguistic example would be a sentence such as "table four wants a big salad," as said by a waitress to another. Here "table four" stands for "the customer at table four." Metonymic mappings occur frequently in mathematical discourse, for example, when we say "the function approaches zero," where "the function" actually stands for "the value of the function."

Embodiment and situatedness

The fact that embodiment specifies a link between cognition and experience which is shared, mostly unconscious, and non-arbitrary, helps to answer our question about the grounding for situated knowing and learning; that is, it offers a rationale for the mutual understanding that is a feature of social situatedness. As Johnson says,

> Meaning is always a matter of human understanding, which constitutes our experience of a common world that we can make sense of. A theory of meaning is a theory of understanding. And understanding involves image-schemata and their metaphorical projections ... These embodied and imaginative structures of meaning have been showed to be shared, public, and 'objective', in an appropriate sense of objectivity (Johnson, 1987, p. 174).

From this point of view, cognition is neither subjective and isolated unique to an individual nor completely determined by external influences. Also, conventionalized meaning, although it is never context-free, depends in a great extent on shared image-schemata and conceptual projections, practices, capacities, and knowledge. Meaning is in many ways socially constructed, but, it is not arbitrary. It is subject to constraints which arise from biological embodied processes that take place in the ongoing interaction between mutually constituted sense-makers and the medium in which they exist (Núñez, 1997). Therefore, it is not surprising that cognition and learning are situated. Cognition is embodied, is biologically grounded, inter-individually, and hence is also social and cultural.

4. A CASE STUDY: CONTINUITY

It is widely accepted that teaching and learning the concept of "continuity" of a function, so important for calculus, is a difficult task (Keisler, 1976; Tall & Vinner, 1981; Robert, 1982 ; Núñez, 1993; Kitcher, 1997). The question then is, Why is this the case? Is continuity *per se* a difficult concept? In this section we would like to illustrate how embodied cognition offers fruitful answers to these questions, and how it provides foundations of situated cognition and learna function $y = f(x)$ is continuous if it displays similar behavior, that is, if a small change in x produces a small change in the corresponding value $f(x)$ ... Up to this stage, our remarks about continuity have been rather loose and intuitive, and intended more to explain than to define (Simmons, 1985).

In the same text, pages further, one finds what is called the "rigorous", "formal" and definitive definition of continuity of a function, namely:

A function f is continuous at a number a if the following three conditions are satisfied:

1. f is defined on an open interval containing a,

2. lim$x$>a $f(x)$ exists, and

3. $\lim x>a\ f(x) = f(a)$.

Where $\lim x>a\ f(x)$ (the limit of the function f at a) is defined as:

Let a function f be defined on an open interval containing a, except possibly at a itself, and let L be a real number. The statement

$\lim x>a\ f(x) = L$

means that for every $e > 0$, there exists a $d > 0$, such that

if $0 < |xa| < d$, then $|f(x)\ \ L| < e$.

This definition of continuity of a function also called Cauchy-Weierstrass definition is said to be, and taught as, the definition that really captures the *essence* of what continuity *is*. It is considered, and taught as, superior and more precise than the so-called "intuitive" and "informal" one. Moreover, using the words of the textbook cited above, this definition intends more to define than to explain.

So far, this is the standard (and from our point of view, misleading, mind-free, non-situated, disembodied) story. Let us step back, and carefully analyze what is going on, cognitively, when one is to understand the statements and ideas involved in the two definitions.

The two definitions of continuity

The informal/intuitive definition that characterizes a "continuous process as one that proceeds without gaps or interruptions or sudden changes" was used by outstanding and creative mathematicians such as Newton and Leibniz in the 17th century, and was characterized by eminent mathematicians such as Euler and Fourier as "a curve described by freely leading the hand". This definition involves cognitive contents such as motion, flows, processes, change in time, and wholeness. These cognitive contents are the result of natural conceptual extensions from bodily grounded image-schemata and conceptual mappings that are natural to the human conceptual system. They are built, among others, on source-path-goal schemata, fictive motion metaphors, and basic conceptual blends (for details see Lakoff & Núñez, 1997). It is precisely because of this that the textbook previously mentioned is perfectly right by referring to this idea as occurring in "everyday" speech. What Euler, Newton, Leibniz, and Fourier did was simply (and, probably, unconsciously) apply the inferential structure of everyday understanding of motion, flow, and wholeness, to a specific domain of human understanding: functions and variations. For the purposes of this study case, we will call this concept *natural continuity*.

The Cauchy-Weierstrass' definition on the other hand, is realized through radically different cognitive content. It implicitly denies motion, flow and wholeness, dealing exclusively with static, discrete, and atomistic elements, which are conceptual extensions of rather different cognitive primitives, such as part-whole schemata, container schemata, and a combination of different conceptual mappings. The point is that, cognitively speaking, these two definitions are simply radically different. Per se, neither of them is better than or superior to the other. Although it is true that the so-called "rigorous" definition deals better with complex and "pathological" cases (such as $f(x) = x \sin 1/x$) for certain purposes, it is not because it captures better the "essence of continuity". It does it simply because it is built on a different collection of bodily grounded conceptual mappings that happen to deal well with both, the prototypical cases of functions mathematicians prior to the 19th century dealt with (e.g., $f(x) = \sin x$; or $f(x) = 1/x$), as well as with the so-called pathological cases (hence its utility and efficacy!) But this, does not have anything to do with cognitive, pedagogical, or philosophical superiorness.

For the purpose of this article, the problem can be summarized as follows: Students are introduced to *natural continuity* using concepts, ideas, and examples that go in line with its inferential patterns sustained by the natural human conceptual system (this does not necessarily mean that they are simple ideas!). Then, they are introduced to another idea that rests upon radically different cognitive contents (although not necessarily more complex), which provides different inferential structure, and different entailments that conflict with those from the previous idea. The problem is that, while doing so, they are never told that the new definition is actually a completely different human embodied idea, and what is worse, they are told that it does capture the essence of the old idea which by virtue of being "intuitive" and vague, is to be avoided. This essence is usually understood as situation-free, that is, independent of human understanding, social activity, and philosophical enterprises.

Backing the claims with in depth analysis

Let us analyze, from the perspective of an embodied cognition why these two concepts of continuity natural and Cauchy-Weierstrass are cognitively so different. Although it is not the scope of this article to show a complete cognitive analysis of these two ideas (for a complete analysis, see Lakoff & Núñez, 1997), we will show here a few relevant aspects for the purpose of a deeper comparison. In particular we will center on the fact that the term "continuity", as used in discussion of mathematics, can mean three distinct ideas. One of them is *natural continuity* (as in the "informal", "intuitive" definition) and the other two (implicit in the "rigorous" definition) are *Gaplessness* (for lines as sets of points) and *Preservation of Closeness* (for functions).

i) Natural continuity. The following are some essential features of a continuous function according to natural continuity:

a) the continuous function is formed by motion, which takes place over time.

b) there is a directionality in the function.

c) the continuity arises from the motion.

d) since there is motion, there is some entity moving, (in Euler's version, the hand).

e) the motion results in a static line with no "jumps."

f) the static line that results has no directionality.

The question is, from where do come motion, static aspects, and directionality? The answer is that we can conceive the mobile and static aspects of the continuous curve via the activation of an everyday human cognitive process: the fictive motion metaphor (Talmy, 1988), which can be summarized as:

A Line IS The Motion of a Traveler tracing that line.

Examples of this mapping are abundant in everyday language:

- Highway 80 *goes to* Sacramento.

- Just before Highway 24 *reaches* Walnut Creek, it *goes through* the Caldecott Tunnel.

- After *crossing* the bay, Highway 80 *reaches* San Francisco.

In these cases a highway, which is a static object, is conceptualized in terms of a traveler moving along the route of the highway. Using the same cognitive mechanism, we can speak, in mathematics, of a function as *moving, growing, oscillating, approaching values,* and *reaching* limits. It is worth noticing that this way of speaking is not limited to students but includes professional mathematicians as well. Formally speaking, the function does not move, but cognitively speaking, under this metaphor, it does - and that is what matters in terms of understanding!

These embodied natural and everyday human cognitive mechanisms are the ones that make possible the intuitive dynamic and static conceptualizations of a continuous function. As in the case of Euler's characterization, the continuity is characterized by motion in the Fictive Motion version. Using this methodology we can give a precise cognitive account of Euler's intuitive notion of continuity for a function in terms of elements of ordinary embodied human cognition, showing how mathematical ideas are constituted out of ordinary bodily grounded ideas.

ii) Cauchy-Weierstrass continuity. This definition is motivated by complex mathematical objects that mathematicians faced in the 19th century, and emerges from three important intellectual movements of that time: the arithmetization of analysis; the set-theoretical foundations movement; and the philosophy of formalism. These movements were separate in their goals, but complementary in their effects on the development of mathematics. All of them required conceptualizing lines, planes, and *n*-dimensional spaces as *sets of points*.

This definition requires a series of cognitive primitives, also embodied in nature, but different from the ones we see for the case of natural continuity. Among others, there are at least three conceptual metaphors that are relevant that combine their inferential structure in a systemic way to give an extremely powerful mathematical tool (again, for a complete description, see Lakoff & Núñez, 1997). These metaphors are:

A Line IS a Set of Points

Natural Continuity IS Gaplessness

Approaching a Limit IS Preservation of Closeness Near a Point

A Line IS a Set Of Points

In general terms, there are two importantly different ways of conceptualizing a line (either a curved or straight line):

1) a holistic one, not made up of discrete elements, where a line is absolutely continuous and points are locations *on* a line. In this sense, a line is an entity distinct from the points, that is, locations on that line, just as a highway is a distinct entity from the locations on that highway. Lines, from the perspective of our everyday geometric intuition, are natural continua in this sense.

2) A Line Is A Set Of Points. According to this metaphor, the points are not locations *on* the line, but they are entities *constituting* the line.

The first characterization is the one that is congruent with natural continuity and the second one with Cauchy-Weierstrass' definition. The distinction between these two ways of conceptualizing lines (and planes and *n*-dimensional spaces) has been crucial throughout the history of mathematics, and the failure to distinguish between them has led to considerable confusion. Both conceptions are natural, in that both arise from our everyday conceptual system. Neither is "right" or "wrong;" however, they have very different cognitive properties, and provide different inferential structure. It is this fact that should be taken into account in the teaching and learning process.

Natural Continuity IS Gaplessness

According to our everyday intuition, a line constitutes a *natural continuum*. It is not conceptualized as made up of points; rather points are conceptualized as *locations* on the line. The line itself is an entity distinct from the point-locations on it. We understand lines, that is, natural continua, without any jumps or gaps, as being continuous. As we move along a line, we go through point-locations. When we move continuously along a line from a location *A* to a location *B*, we go through all point-locations on the line between *A* and *B*, without skipping over any, that is, without leaving any gaps between the point-locations. In this case we will say that the collection of point-locations between *A* and *B* is *gapless* when the line segment *AB* is naturally continuous.

This metaphor identifies the point-locations on a line, that is, a natural continuum, as constituting the line itself. Such a metaphorical "line" *is not* a natural continuum, but only a set of points. Given a naturally continuous line segment *AB*, the point-locations on that line will be *gapless*. Similarly, when a naturally continuous line segment is conceptualized as a set of points, that set of points will be *gapless*. Thus, in this specific situated conceptual context the metaphor A Line Is A Set Of Points entails the metaphor:

Natural Continuity IS Gaplessness.

Therefore, a line conceptualized as a set of points cannot be cognitively *naturally continuous* but only *gapless*. This terminology distinguishes thus two distinct ideas, based on different cognitive mechanisms, that have previously both been called "continuity."

Approaching a Limit IS Preservation of Closeness Near a Point

In Cauchy-Weierstrass' definition of limit there is no motion, no time, and no "approach." Instead, there are static elements. Besides, the definition calls for a *gapless* "open interval" of real numbers; there are no lines and no points and no surfaces in that metaphorical ontology for the Cartesian plane. The plane itself is a made up of a set of pairs of real numbers. The gaplessness of the set of real numbers in the open interval in the definition is Cauchy-Weierstrass' metaphorical version replacing the natural continuity of the intuitive line in Newton's geometric idea of a limit.

The idea of the function *f* approaching a limit *L* as *x approaches a* is replaced by a different idea (in order to arithmetize avoiding motion), that is, *preservation of closeness near a real number: f(x)* is arbitrarily close to *L* when *x* is sufficiently close to *a*. The epsilon-delta condition expresses this precisely in formal logic. What the Cauchy-Weierstrass approach does is to give a new metaphor:

Approaching A Limit IS Preservation Of Closeness Near A Point.

When Cauchy-Weierstrass "define" continuity for a function, they cannot mean cognitively the natural continuity assumed by Newton for ordinary lines, that is, natural continua. Again, they must use conceptual mappings (metaphors) that allow them to reconceptualize geometry (holistic lines) using arithmetic (discrete numbers). Just as they needed a new metaphor for approaching a limit, they needed a new metaphor for continuity of a function. They characterize this new metaphor in two steps: first at a single arbitrary real number and then throughout a (gapless) interval. Their new metaphor for continuity uses the same basic idea as their metaphor for a limit: *preservation of closeness.* Continuity at a real number is conceptualized as preservation of closeness not just near a real number but also *at* it. Continuity of a function throughout an interval is thus preservation of closeness near and at every real number in the interval.

What is precise in the Cauchy-Weierstrass definition?

Textbooks and curricula lead students (and nearly everybody in the mathematical community) to believe that it is the epsilon-delta portion of these definitions that constitutes the rigor of this arithmetization of analysis. Moreover, they are led to believe that it is this aspect that helps to capture the essence of what "continuity" *is*. As we saw, not only this is not true, but also, the epsilon-delta aspect of the definition actually plays a far more limited role. The epsilon-delta aspect accomplishes only a precise characterization of the notion "correspondingly." But this notion can be included in the dynamic definition of a limit aas well, where the values of $f(x)$ get "correspondingly" closer to L as x gets closer to a.

Another interesting element in the Cauchy-Weierstrass definition is the role played by the idea of "gaplesness". Cauchy-Weierstrass formulate the "definition of continuity" with the explicit condition that the function is defined over an open interval. It assumes this open interval to be gapless. Since gaplessness was a barrier to metaphorically conceptualizing continuity on the real line, it assumes a "continuous," that is, gapless input to the function. What this definition really shows is that (1) when these metaphors hold, especially when lines are metaphorically conceptualized as sets of real numbers, and (2) when the input of the function is gapless, and (3) when the function preserves closeness, then (4) the output is also gapless.

Why has it been widely accepted that Cauchy-Weierstrass' definition of preservation of closeness was instead a "definition of continuity"? The answer is that it has been assumed, falsely, that Cauchy-Weierstrass' metaphors capture the essence of continuity because it effectively deals, for the purposes of the arithmetization program, with prototypical and pathological cases. Given the metaphor that a line is a set of real numbers, then natural continuity can only be conceptualized metaphorically as gaplessness. Since Cauchy-Weierstrass' open interval condition guaranteed that the inputs to the function are always gapless, it is no surprise that preservation of closeness for a function with a gapless input guarantees a gapless output. If the input is metaphorically continuous (that is, gapless), then the output is going to be metaphorically continuous (gapless). Since the metaphors are not noticed as being metaphorical or controversial in any way (they are mostly realized through unconscious processes), and since the open interval condition hid the continuity (gaplessness) required in the input, Cauchy-Weierstrass' definition appeared even to the originators to be a definition of continuity, when in fact, all it did was guarantee that a gapless input for a function gives a gapless output.

5. DISCUSSION

As a discipline, mathematics education is concerned not only with creating effective means and methods of instruction, but with understanding why certain methods are effective and others are not, and with larger questions about the nature and development of mathematical knowledge. The answers to these questions, and even the ways one chooses to investigate them, are strongly influenced by the implicit or explicit conceptualization about the nature of human thought, and about mathematics itself. When mathematics is conceived of as an external realm of objective truths, to be "discovered" through the application of rational thinking, then the investigation of mathematics learning focuses on accurate mappings, models, or internal representations of mathematical entities or relationships. If, on the other hand, mathematics is conceived as a product of adaptive human activity in the world, shared and made meaningful through language, but based ultimately on biological and existential experiences unique to our species, then mathematics education must take a different approach. New forms of doing mathematics education, from classroom practice to scientific research to curriculum design, should emerge such that mathematics is seen and taught as a genuine mind-based activity with all its embodied peculiarities and beauty.

Through the analysis of the idea of continuity, we have seen that certain ways of talking and thinking about mathematics can be misleading, with unfortunate pedagogical consequences. These consequences can arise when we ignore how our conceptual system works, and by implicitly assuming the existence of a "mind-free" mathematics. We propose that an important source of pedagogical problems in mathematics education are the philosophical foundations underlying the view of mathematics that dominates our culture (objectivism, Platonism, formalism). These philosophical commitments are necessarily (if unintentionally) transmitted in the teaching process, which can lead to the teaching of supposed eternal, timeless, truths that capture essences, rather than mind-based, embodied, human forms of sense-making. The fundamental conceptual error underlying this kind of teaching is the idea that intuition can be replaced by rigor in order to eliminate vagueness. Not only is this not possible (and not actually necessary, c.f., Smith, diSessa, and Roschelle, 1993/94), but if one studies natural, situated, spontaneous, everyday thoughts and intuitions from an embodied cognitive perspective, one finds that intuitions are not at all vague. The tools provided by the embodied cognition approach allow one to characterize precisely how the inferential structure of everyday bodily experience, which underlies intuition, is mapped onto other domains. This process results in more abstract understandings which still preserve the originial image-schematic inferential structures.

Basic mathematical ideas show an impressive stability over hundreds, sometimes thousands of years. For this to happen requires, on the one hand, a common set of extremely basic (although complex) neural and bodily structures through which the mathematical ideas are realized. On the other hand, it requires that the conceptual construction of mathematics make use of the most commonplace of everyday experiences, such as motion, spatial relations, object manipulation, space, and time. The study of the conceptual structure of mathematics from an embodied point of view shows how mathematics is built up out of such informal, everyday experiences and ideas. For this reason, mathematics is not a pure and "abstract" discipline, as represented by influential schools of thought within our culture. Our mathematical conceptual system, like the rest of our conceptual system, is grounded in our shared bodily functioning in the world, in our very bodily experiences. Therefore, seen from this perspective, situated cognition is not about "situating" mind-free truths in meaningful contexts, but rather about seeing how the human creation of mathematics arises from human sense-making that is not arbitrary because it is bodily grounded.

This view has important entailments for mathematics education. Rather than looking for ways to provide students with correct understandings of a pre-given mathematics, we need to examine the kinds of understanding and sense-making we want students to develop. First, we should look at the everyday experience that provide the initial grounding for abstractions that constitutes the human intellectual construction known as mathematics. This is not necessarily an easy undertaking, since the grounding structures are often completely unconscious and taken-for-granted. At times, this grounding can be found in the immediate physical experience of teachers and learners, as in the case of early work with arithmetic, space, size, and motion. At other times, the grounding for a mathematical idea takes place indirectly, through a chain of conceptual mappings whose nature may be obscured by conventional language, but which can be revealed and studied utilizing the analytic tools of contemporary embodied cognitive science. In either case, what is most important is to create learning environments that complement the ways our conceptual systems naturally work, by extending bodily grounded natural sense-making.

Second, we should also design mathematical pedagogy which talks about the discipline as one aspect of human sense-making. In such an approach, the notion that mathematics knowledge is about eternal truths is refuted. Students should know that mathematical theorems, proofs, and objects are about ideas, and that these ideas are situated and meaningful because they are grounded in our bodily experience as social animals.

Third, we should also provide a learning environment in which mathematical ideas are taught and discussed with all their human embodied and social features. This means that in order to really make sense of mind-based mathematical ideas as fully embodied and contextual, one needs also to provide an understanding of the historical processes through which embodied ideas emerged. This does not mean simply presenting a few names and dates as a prelude to teaching the "real" mathematics. It means talking about the motivations, zeitgeist, controversies, difficulties, and disputes that motivated and made possible given mathematical ideas. For example, in the case of continuity, Pierpont (1899) provides excellent material for appreciating the controversies surrounding intuition and the arithmetization program at the turn of the century.

Finally, we should develop new programs for mathematics educators so that they can become sensitive to the richness and beauty of how our everyday conceptual systems work. For a number of years, research and teacher education have addressed students' errors and "misconceptions," acknowledging at least that there is a logic and systematicity to students' first ways of thinking about mathematics and science. We need to go beyond the identification of students' naive notions to understand how this thinking arises, and how it can form a foundation for further conceptual extensions to more abstract structures (Smith et. al., 1993/94). We should encourage new teachers to become aware of the advances of the study of everyday embodied cognition, and its important role in the grounding of mathematical ideas.

In this paper, we have attempted to provide an overview of the essential elements of a theory of embodied cognition, and to apply this relatively new framework to the analysis of mathematical thought and learning. This framework is extremely rich, and methodologies and applications of the embodied cognition approach are still emerging, with systematic work on the analysis of mathematical thought only beginning to take place. An abbreviated account of one case of such an analysis was provided, in order to illustrate the potential of the framework for enhancing our understanding of how mathematics is learned, and why it is, sometimes, not learned in the ways we hope. In addition to discussing particular ways in which mathematical thought arises from embodied experience, we also addressed the relationship between theories which emphasize the socially-situated nature of cognition and embodied cogntions approaches. From our perspective, there is no basic contradiction between these approaches; rather, an understanding of the fundamental embodiment of cognition helps us to see how human beings are able to construct mutual understandings by means of social interaction. Since we are all living physical creatures, part of the same medium (or environment), and sharing a basic biological heritage, we naturally experience the world in fundamentally similar ways. The conceptual structures which emerge in the human mind to make sense of our bodily experiences provide the raw material for the construction of shared communication through language, and, subsequently, the shared construction of meanings. Thus, our sense-making of the world, and of mathematics, may be socially and culturally situated, but is the commonalties in our physical embodiment and experience that provide the bedrock for this situatedness.

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