

# PART 2

# THE GOALS OF MATHEMATICAL EDUCATION<sup>1</sup>

George Polya

(As transcribed by Thomas C. O'Brien from a videotaped lecture. The video – 30 plus-years-old – was transcribed thanks to the painstaking technical work of John Ruiz and Steve Berkemeier.)

Take as much of my advice as it fits you personally. You must teach yourself.

The most important action of learning is to discover it by yourself. This will be the most important part in teaching.

For part one of this lecture see page 6 of this issue.

Teaching is not a science; it is an art. If teaching were a science there would be a best way of teaching and everyone would have to teach like that. Since teaching is not a science, there is great latitude and much possibility for personal differences. In an old British manual there was the following sentence, 'Whatever the subject, what the teacher really teaches is himself.' So therefore when I am telling you to teach so or so, please take it in the right spirit. Take as much of my advice as it fits you personally. You must teach yourself.

There are as many good ways of teaching as there are good teachers. But let me tell you what my idea of teaching is. Perhaps the first point, which is widely accepted, is that teaching must be active, or rather active learning. That is the better expression.

You cannot learn just by reading. You cannot learn just by listening to lectures. You cannot learn just by looking at movies. You must add from the action of your own mind in order to learn something. You can call this the Socratic method since Socrates expressed it two thousand years ago very colourfully. He said that the idea should be born in the student's mind and the teacher should just act as a midwife. The idea should be born in the student's mind naturally and the midwife shouldn't interfere too much, too early. But if the labour of birth is too long, the midwife must intervene. This is a very old principle and there is a modern name for it – discovery method. The student learns by his own action. The most important action of learning is to discover it by yourself. This will be the most important part in teaching such that what you discover by yourself will last longer and be better understood.

There are other principles of teaching. If you don't like the word *principles*, use *rules of thumb*. Learning should be active. Another one was also stated often by all great famous educators – by Socrates, Plato, Comenius, Montessori – and that is that there are certain priorities. For instance, things come before words and so on. This has been stated many times in many forms, but let me quote Kant, who said, 'All human cognition begins with intuitions, proceeds hence to conceptions, and ends in ideas.' Let me translate this saying into simpler terms. I would say, 'Learning begins with action and perception, proceeds hence to words and concepts, and should end in good mental habits.'

This is the general aim of mathematics teaching – to develop in each student as much as possible the good mental habits of tackling any kind of problem.

You should develop the whole personality of the student and mathematics teaching should especially develop thinking. Mathematics teaching could also develop clarity and staying power. It could also develop character to some extent but most important is the development of thinking.

My point of view is that the most important part of thinking that is developed in mathematics is the right attitude in tackling problems, in treating problems. We have problems in everyday life. We have problems in science. We have problems in politics. We have problems everywhere. The right attitude to thinking is maybe slightly different from one domain to another, but we have only one head, and therefore it is natural that finally there should be just one method of tackling all kinds of problems. My personal opinion is that the main point in mathematics teaching is to develop the tactics of problem-solving.

The two principles of active learning – priority

of action and perception – are taken into account by almost all directions in mathematics teaching which are usual today and have some influence. Perhaps the best developed in the latest time is in Great Britain. There is a foundation called the Nuffield Foundation, which propagates this idea of active learning and the priority of action and perception in learning. Their first book has a nice motto. It is allegedly a Chinese proverb that says, ‘I hear and I forget. I see and I remember. I do and I understand.’ [1]

So ‘I hear and I forget.’ What you just hear you forget quickly. Good advice is very quickly forgotten. What you see with your own eyes is remembered better, but you really understand it when you do it with your own hands. So the motto is ‘I hear and I forget. I see and I remember. I do and I understand.’

Therefore the schools, especially the primary schools, are today in an evolution. A sizable fraction, ten to twenty percent, already have the new method of teaching which can be characterized in the following way in comparison with the old method of teaching. The old method is authoritative and teacher-centered. The new method is permissive and student-centered. In the old time the teacher was in the centre of the class or in front of the class. Everybody looked at him and what he said. Today the individual students should be in the centre of the class, and they should be allowed to do whatever good idea comes to their mind. They should be allowed to pursue it in their own way, each by himself or in small groups. If a student has a good idea in class discussion then the teacher changes his plans and enters into the good idea and now the class follows this idea.

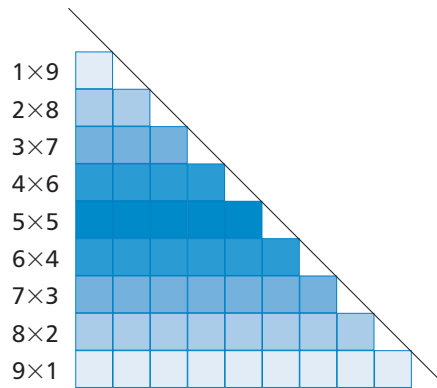
I must tell you one name. This is the person who is particularly active in this direction and who is very clever, very good. This is Miss Edith Biggs. She is a particularly gifted teacher who stands in with great enthusiasm and talent for this new permissive and student-centered teaching<sup>2</sup>.

In such a permissive and student-centered class, each group of kids do something else. They play (let’s just say that they think that they play, but really they learn). The teacher gives them various materials. A class period consists of the teacher giving kids various materials and they play and they develop their own ideas in play. For instance, one of the materials is squared paper. And a good supply of cubes, cubes of one half inch and several dozens of them, maybe even a hundred. So the kids play with that. It is activity teaching – teaching by action and perception.

## An example

Let me give you an example of this activity. The class discusses little rectangles. It should come – that’s the main point – from action and perception. It should come from things which kids have seen often enough and touched. So everybody has seen a room, and the walls of an ordinary room are rectangles, or almost rectangles. So you learn what a rectangle is. The floor of the usual room is a rectangle. And any wall is a rectangle. The ceiling is a rectangle. One of the good aims of teaching, then, is to understand length and area. So you measure the length of the rectangles and come to the idea of the perimeter of the rectangles. Then you deal with the area of the rectangle. You build up the rectangle from equal squares, from unit squares, and come to the notion of the area. Anyway, we are now in a class that is somewhat familiar with the area and perimeter of rectangles.

On the same sheet of paper, draw overlapping rectangles, with the same perimeter – a perimeter of twenty. It turns out that there are nine such rectangles. They start with width = 1 and height = 9, and then width = 2 and height = 8, and down to width = 9 and height = 1.



There are many things to observe – action and perception. Some of the kids will be struck by the observation that all the corners of these rectangles are on a straight line. Then they will notice that one of these rectangles has equal sides and you can ask many questions about it. One of the interesting points is that the teacher should not ask the questions but the kids should ask the questions. They all have the same perimeter. Do they have the same area? Which one has the greatest area?

Here is another activity with rectangles. Again take square papers and cut out different rectangles with the same areas, let’s say area of 24 square units. Overlap them on the same paper. Now the corners opposite to the one corner in which they overlap are not on a straight line. There is some funny kind of curved line.

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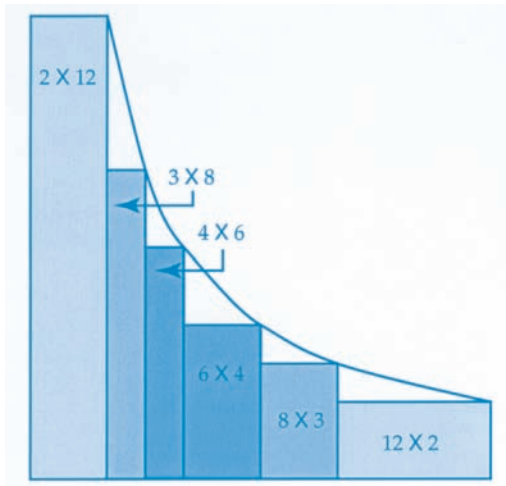
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**Footnotes**

- 1 Part II of this transcribed lecture appeared in the December 2001 issue of *ComMuniCator*, the journal of the California Mathematics Council, Volume 26.2.
- 2 An obituary to Edith Biggs was published in MT180, September 2002

**Reference**

- 1 Nuffield Mathematics Project: *I do, and I understand*; published for the Nuffield Foundation by W&R Chambers and John Murray, London, 1967



Kids with an imagination will join these to make curved lines. So that is another consideration. This is an example of an activity with rectangles where the kids have their own choice. They make their own remarks and the teacher just helps a little now and then with some hints. If the kids have no ideas at all, then the well instructed teacher, who is used to this student-centered teaching, can give a few good hints.

Perhaps one point which Miss Biggs and the Nuffield Foundation do not emphasize sufficiently is the rule of guessing. Guessing comes to us naturally. Everybody tries to guess and does not have to be taught. What has to be taught is reasonable guessing. And especially what has to be taught is not to believe your own guesses but to test them. And

students' activity will start much better if you start them by guessing.

Here is one example. One activity is to measure the length and the width of the classroom. Now some kids may be bored by this if they already did it with a former teacher. You can get a little more attention if you start with a guess. You may say, 'It seems to me that this classroom is twice as long as it is wide. Is it really?' I hope some of the kids will say, 'No, it is longer than twice.' Others will say, 'No, it is shorter'. A very few will say, 'Exactly'. After they have guessed, they will do the measuring with much more interest because everybody is interested whether his guess will come true or not. This is a very special case in the tactics of problem-solving. If you go farther, you will notice that guessing plays an important role. The solution to a problem naturally starts always with a guess – not always with a good guess. On the contrary, usually the guess is never completely good. It is just a little out of centre and the art of problem-solving consists in great part in correcting your guesses.

I have given you my ideas about how you should teach mathematics. There are the ideas of active learning, the priority of action and perception, and teaching by the activity of the kids to start them by letting them guess. I hope one of these points will find a sympathetic hearing with some of you. Thank you.

*George Polya (1887-1985) was a mathematician and professor at Stanford University, USA.*

## PAPER FOLDING FRACTIONS

In MT180 Malcolm Swan described an ingenious way to divide up the length of a rectangle into  $1/2, 1/3, 1/4, \dots$  in turn, using a series of folds.

The diagram shows one step in the process, which started with the rectangle ABCD folded along one diagonal AC. Assume that we have at this stage found P which divides AB in a known ratio.

Two more folds determine the point Q on AB.

The first is along PD, cutting AC at X; the second is through X at right angles to AB (ie, folding this line along itself), cutting AB at Q. Malcolm says that if  $AP = (1/n) AB$ , then  $AQ = (1/(n+1)) AB$ , and asks for a proof.

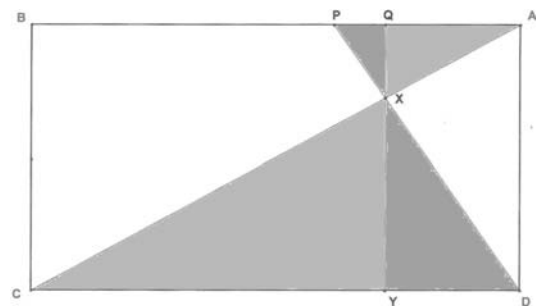
I would like to restate this as 'the ratio  $AQ:QB$  is equal to the ratio  $AP:AB$ '.

So for example if  $AP = (2/5) AB$  then  $AQ:QB = 2:5$ , so that  $AQ = (2/7) AB$ .

More generally if  $AP = (p/q) AB$  then  $AQ = (p/(p+q)) AB$ . Malcolm has  $p=1, q=n$ .

To prove this, use similar triangles and the fact that opposite sides of a rectangle are equal.

$$\text{So } \frac{AP}{AB} = \frac{AP}{CD} = \frac{AX}{XC} = \frac{AQ}{CY} = \frac{AQ}{QB}$$



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