The method[1]

Hans Freudenthal, *University of Utrecht*

2.1. Aspects of Phenomenology

I started with an example to be used as a subject matter which I can appeal to when I explain my method. I chose "length" because it is both a rich and relatively easy subject.

First of all, what of the terms "phenomenology" and "didactical phenomenology"? Of course I do not mean "phenomenology" in the sense that might be extracted from the works of Hegel, Husserl, and Heidegger[2]. Though the clearest interpretation I can imagine is that by means of the example of chapter I, which is to be continued in the following chapters, nevertheless it is worthwhile trying something like a definition.

I start with the antithesis – if it really is an antithesis – between *nooumenon* (thought object) and *phainomenon*. The mathematical *objects* are *nooumena*, but a piece of mathematics can be experienced as a *phainomenon*; numbers are *nooumena*, but working with numbers can be a *phainomenon.*

Mathematical concepts, structures, and ideas serve to organise phenomena – phenomena from the concrete world as well as from mathematics – and in the past I have illustrated this by many examples[3]. By means of geometrical figures like triangule, parallelogram, rhombus, or square one succeeds in organising the world of contour phenomena; numbers organise the phenomenon of quantity. On a higher level the phenomenon of geometrical figure is organised by means of geometrical constructions and proofs, the phenomenon "number" is organised by means of the decimal system. So it goes in mathematics up to the highest levels: continuing abstraction brings similar look-

[1] Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures*. Dordrecht: D. Reidel. (Capítulo 2)
[2] Is it by accident that – with Habermas included – the names of the most pretentious producers of unintelligible talk in the German philosophy start with an H?
[3] *Mathematics as an Educational Task*, in particular, Chapters II and XVII.

ing mathematical phenomena under one concept – group, field, topological space, deduction, induction, and so on.

Phenomenology of a mathematical concept, a mathematical structure, or a mathematical idea means, in my terminology, describing this *nooumenon* in its relation to the *phainomena* of which it is the means of organising, indicating which phenomena it is created to organise, and to which it can be extended, how it acts upon these phenomena as a means of organising, and with what power over these phenomena it endows us. If in this relation of *nooumenon* and *phainomenon* I stress the didactical element, that is, if I pay attention to how the relation is acquired in a learning–teaching process, I speak of *didactical* phenomenology of this *nooumenon*. If I would replace "learning–teaching process" by "cognitive growth", it would be *genetic* phenomenology and if "is … in a learning–teaching process" is replaced by "was … in history", it is *historical* phenomenology. I am always concerned with phenomenology of mathematical nooumena, although the terminology could be extented to other kinds of nooumena.

2.2. The Part Played by Examples

The piece of phenomenology with which Chapter I began was clearly an *a posteriori* constructed relation between the mathematical concept of length and the world of long objects structured by an operation of composing, ? . Length was interpreted as a function on this world. I did not analyse how I arrived at this function. Although this was indispensable, I omitted it because I had to tackle this question in the didactical phenomenological section and I wanted to avoid duplication. But as a consequence the didactical phenomenological section contains pieces of pure phenomenology, such as Section 1.15 about the congruence mappings and Sections 1.18–19 about the flexions. Likewise in the sequel I will not clearly separate phenomenology and didactical phenomenology from each other. As promised in the preface I would not sacrifice readability to systematics.

Where did I look for the material required for my didactical phenomenology of mathematical structures? I could hardly lean on the work of others. I have profited from my knowledge of mathematics, its applications, and its history. I know how mathematical ideas have come or could have come into being. From an analysis of textbooks I know didacticians judge that they can support the development of such ideas in the minds of learners. Finally, by observing learning processes I have succeeded in understanding a bit about the actual processes of the constitution of mathematical structures and the attainment of mathematical concepts. A bit – this does not promise much, and with regard to quantity it is not much, indeed, that I can offer. I have already reported a few examples of such observations, and I will continue in the same way. I do not pretend that at this or that age this or that idea is acquired in this or that way. The examples are rather to show that learning processes are required for things which we would not expect would need such processes. In the first chapter I showed a child suddenly confronted with the necessity to differentiate "big" according to various dimensions, a child placing "far" into the context of "long" and learning about the connection between "half" and "middle". I am going to add another story, which happened a few hours after the event where "half" and "middle" were tied to each other:

> Bastiaan's (5; 3) sister (3; 3) breaks foam plastic plates into little pieces, which she calls food. He joins her, takes a rectangular piece, breaks it in about two halves, lays the two halves on each other, breaks them together and repeats the same with a three-layered combination – the fourth piece was already small enough.

I do not know where I should place this observation, whether I should classify it as mathematics, say geometry, or whether it belongs to general cognitive behaviour. I report this observation because I think it is one of the most important I ever made because it taught me a lesson on observing. I do not know whether the age of 5; 3 is an early or a late date for this kind of economic breaking; I do not know either whether Bastiaan imitated or adapted something he had observed before. I know only one thing for sure: that what he did is important and worth being learned. For myself it is fresh material to witness that in no way do we realise all things children must learn. If I look at what people contrive to teach children, I feel inclined to call out to them: do not exert yourself, simply look, it is at your hand.

Why do people not look for such simple things, which are so worth being learned? Because one half of them do not bother about what they think are silly things, whereas those who do bother are afraid to look silly themselves if they show it. *Weeding and Sowing* is full of such simple stories. I told them in lectures. I do not care whether a large part of the audience interprets my reporting as senility, provided that by my example a small part of the audience is encouraged to follow suit – this, indeed, requires courage.

2.3. Enactive, Ikonic, Symbolic

Above I used Bruner's triad "enactive, ikonic, symbolic". Bruner[4] suggested three ways of transforming experiences into a model of the world: the enactive, the ikonic, and the symbolic representation. Corresponding to the dominance of one of these, he distinguishes phases of cognitive growth.

Bruner's schema can be useful. It has been taken over by others, and its domain of application has been extended, in particular towards the attainment of concepts in learning processes, where similar phases are distinguished. Later I will explain my objections to the idea of concept attainment as such, although I would not oppose the extension of Bruner's triad to concept attainment. As a matter of fact, in Bruner's work there is an example that shows how the three ways of representation can be extended to concept attainment: *enactively* the clover leaf knot is a thing that is knotted, *ikonically* it is a picture to be looked at, and *symbolically* it is something represented by the word "knot", whether or not is accompanied by a more or less stringent definition.

There is a well-known pleasantry: ask people what "spiral" stairs are. All react the same way: they make their forefinger mount imaginary spiral stairs. Of course, if need be, they would be able to draw them. Does this mean that they are in the enactive or in the ikonic phase? Of course not. For the concept in question they possess a symbol, the words "spiral stairs", though if a *definition* is to be produced, one would have more or less difficulty in passing from the enactive or ikonic to the symbolic representation.

Consider the number concept "three" and the geometrical concept "straight". Before the child masters these words, he can be familiar with what they mean: clapping his hands thrice and running straight to a goal if it is suggested to him (the enactive phase); sorting out cards with three objects or straight lines pictured on them (the ikonic phase). Mastering the word *three* (or *straight)* means he is in the symbolic phase, since "three" as a word is a symbol for the concept three (or "straight" is for *straight*). But likewise the three dots on dice can be a symbol; for instance, in playing the game of goose. A child that counts intelligently is in the symbolic phase even if this counting is accompanied by moving counters on the abacus. Adding on the abacus is enactive only for a

[4] *Studies in Cognitive Growth* (Edited by J. S. Bruner), Toward a Theory of Instruction, 1966, pp. 10-11.

moment. After the first experience it has become symbolic, though the symbolism differs from that of the written digits. The Roman numerals are as symbolic as the Arabic ones. Notches and tallies to indicate numbers belonged to the symbolic phase, even before people invented numerals – they are as symbolic as Roman and Arabic numerals. The cashier in the supermarket who prints amounts of money is neither enactively nor ikonically busy. A little child who claps his hands in joyfulness expresses his feelings symbolically even if he cannot yet pronounce the word joy. As early as kindergarten, children accept a drawing of a dance position where dancers are represented by strokes rather than manikins. If the doors of the men's and ladies's rooms are distinguished by plates of figures in trousers and skirts it does not mean that the decorator imagined the users to be in the ikonic phase; he did so because this difference is differently symbolised in the hundreds of languages that mankind speaks and writes – moreover the plates themselves are already symbols.

With these examples I intend to say that in learning–teaching situations, which are our main interest, Bruner's triad does not yield much. Bruner's domain of application is the psychology of the very young child, and in this period the phases can meaningfully be filled out.

2.4–5. Concept Attainment and the Constitution of Mental Objects

2.4. I would like to stress another idea, already stressed in my earlier publications. Let me start with a semantic analysis of the term "concept". If I discuss, say, the number concept of Euclid, Frege, or Bourbaki, I set out to understand what these authors had in mind when they used the word "number". If I investigate the number concept of a tribe of Papuans, I try to find out what the members of this tribe know about and do with numbers; for instance, how far they can count.

It seems to me that this double meaning of "concept" is of German origin. The German word for concept is *Begriff* which etymologically is a translation of Latin "conceptus" as well as "comprehensio" and which for this reason can mean both "concept" and "(sympathetic) understanding". "Zahlbegriff" can thus mean two things, number concept and understanding of number; "Raumbegriff", concept of space and geometrical insight; "Zunstbegriff," concept of art and artistic competence.

Actually, in other languages too "concept" is derived from a word that means understanding (English, *to conceive*; French, *concevoir*) which, however, does not have the misleading force that the German *begreifen* has. I cannot say whether it has been the influence of German philosophy – in particular, philosophy of mathematics – that created the double meaning of number concept, of space concept, and in their train as it were, of group concept, field concept, set concept, and so on. At any rate the confusion has been operational for a long time and has been greatly reinforced by the New Math and by a rationalistic[5] philosophy of teaching mathematics (and other subjects) which in no way is justified by any phenomenology. It is the philosophy and didactics of concept attainment, which, of old standing and renown, has gained new weight and authority in our century thanks to new formulations. In the socratic method as exercised by Socrates himself, the sharp edges of concept attainment had been polished, because in his view attainment was a re-attainment, recalling lost concepts. But in general practice the double meaning of concept has been operational for a long time. Various systems of structural learning have only added a theoretical basis and sharp formulations. In order to have some *X* conceived, one teaches, or tries to teach, the concept of *X.* In order to have numbers, groups, linear spaces, relations conceived, one instills the concepts of number, group, linear space, relation, or rather one tries to. It is quite obvious, indeed, that at the target ages where this is tried, it is not feasible. For this reason, then, one tries to materialise the bare concepts (in an “embodiment”). These concretisations, however, are usually false; they are much too rough to reflect the essentials of the concepts that are to be embodied, even if by a variety of embodiments one wishes to account for more than one facet. Their level is too low, far below that of target concept. Didactically, it means the cart before the horse: teaching abstractions by concretishing them.

What a didactical phenomenology can do is to prepare the converse approach: starting from those phenomena that beg to be organised and from that starting point teaching the learner to manipulate these means of organising. Didactical phenomenology is to be called in to develop plans to realise such an approach. In the didactical phenomenology of length, number, and so on, the phenomena organised by length, number, and so on, are displayed as broadly as possible. In order to teach groups, rather than starting from the group concept and looking around for material that concretises this concept, one shall look first for phenomena that might compel the learner to constitute the mental object that is being mathematised *by* the group concept. If at a given age

[5] In the 18th century sense of *a priori* concepts epistemology.

such phenomena are not available, one gives up the – useless – attempts to instill the group concept.

For this converse approach I have avoided the term *concept attainment* intentionally. Instead I speak of the constitution of mental objects,[6] which in my view precedes concept attainment and which can be highly effective even if it is not followed by concept attainment. With respect to geometrically realizable mental objects (square, sphere, parallels) it is obvious that the constitution of the mental object does not depend at all on that of the corresponding concept, but this is equally true for those are not (or less easily) geometrically realizable (number, induction, deduction). The reader of this didactical phenomenology should keep in mind that we view the nooumena primarily as mental objects and only secondarily as concepts, and that it is the material for the constitution of mental objects that will be displayed. The fact that manipulating mental objects precedes making concepts explicit seems to me more important than the division of representations into enactive, ikonic, and symbolic. In each particular case one should try to establish criteria that ought to be fulfilled if an object is to be considered as mentally constituted. As to "length" such conditions might be.

integrating and mutually differentiating the adjectives that indicate length,
with "long, short",
comparing lengths by congruence mappings and flexions,
measuring lengths by multiples and simple fractions of a measuring unit,
applying order and additivity of measuring results, and
applying the transitivity of comparing lengths.

2.5. In opposition to concept attainment by concrete embodiments I have placed the constitution of mental objects based on phenomenology. In the first approach the concretisations have a transitory significance. Cake dividing may be forgotten as soon as the learner masters the fractions algorithmically. In contradistinction to this approach, the material that serves to mentally constitute fractions has a lasting and definitive value. "First concepts and applications afterwards" as it happens in the approach of concept attainment is a strategy that is virtually inverted in the approach by constitution of mental objects.

[6] Fischbein calls them *intuitions*, a word I try to avoid because it can mean inner vision as well as illuminations.