FUNCTIONAL REASONING AND THE INTERPRETATION
OF CARTESIAN GRAPHS

by

JOÃO PEDRO MENDES DA PONTE
Lic., University of Lisbon, 1979

A Dissertation Submitted to the Graduate Faculty
of the University of Georgia in Partial Fulfillment
of the
Requirements for the Degree

DOCTOR OF EDUCATION

ATHENS, GEORGIA
1984
FUNCTIONAL REASONING AND THE INTERPRETATION OF CARTESIAN GRAPHS

by

JOÃO PEDRO MENDES DA PONTE

Approved:

James W. Wilson Date 5/28/84
Major Professor

Sigi Schwalbe Date 5/28/84
Chairman, Reading Committee

Approved:

John Dowling

Date 4/28/84
Graduate Dean
ACKNOWLEDGMENT

This dissertation grew out of the stimulating environment provided by the Department of Mathematics Education of the University of Georgia. I am very grateful to all its faculty and students for sharing their ideas, projects, expertise, and warmth.

I am greatly indebted to Dr. Joseph Hooten for his willingness to hear my concerns, encouragement in pursuing my project, and sound advice. I am also grateful to Drs. Sigrid Wagner and Jeremy Kilpatrick for their advice, support, and criticism that helped to focus this dissertation on significant educational issues and to improve the quality of the reporting.

I am grateful to all high school and college students, teachers, and administrators who cooperated in the execution of this project. I am also grateful to those who at the University contributed in several ways to the successful completion of this study.

My graduate studies at the University of Georgia were made possible by a grant from INVOTAN--Junta Nacional de Investigação Científica e Tecnológica, Lisbon, Portugal, and a leave of absence from the University of Lisbon.
A special word goes to my family. Ana, Joana, Sofia, and Inês were a major source of inspiration, support, and personal encouragement throughout my graduate studies at the University of Georgia.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Acknowledgment</td>
<td>1</td>
</tr>
<tr>
<td>List of Tables</td>
<td>viii</td>
</tr>
<tr>
<td>Chapter 1: FUNCTIONAL REASONING AND GRAPH COMPREHENSION IN SCHOOL MATHEMATICS</td>
<td>1</td>
</tr>
<tr>
<td>Functional Reasoning and Graph Comprehension as Educational Objectives</td>
<td>1</td>
</tr>
<tr>
<td>Students' Difficulties in Functional Reasoning and Graph Comprehension</td>
<td>4</td>
</tr>
<tr>
<td>Chapter 2: THEORETICAL FOUNDATIONS</td>
<td>7</td>
</tr>
<tr>
<td>Mathematical Thinking Processes</td>
<td>7</td>
</tr>
<tr>
<td>Operations and Structures</td>
<td>12</td>
</tr>
<tr>
<td>Conceptual Structures and Understanding</td>
<td>13</td>
</tr>
<tr>
<td>Imagery</td>
<td>15</td>
</tr>
<tr>
<td>Intuitions</td>
<td>17</td>
</tr>
<tr>
<td>Theoretical Framework and Definitions</td>
<td>20</td>
</tr>
<tr>
<td>Chapter 3: REVIEW OF RESEARCH</td>
<td>24</td>
</tr>
<tr>
<td>Research on Functional Reasoning</td>
<td>24</td>
</tr>
<tr>
<td>Research on Graph Comprehension</td>
<td>41</td>
</tr>
<tr>
<td>Chapter 4: METHOD</td>
<td>49</td>
</tr>
<tr>
<td>Sample</td>
<td>50</td>
</tr>
<tr>
<td>Instruments</td>
<td>53</td>
</tr>
<tr>
<td>Procedure</td>
<td>62</td>
</tr>
<tr>
<td>CHAPTER</td>
<td>Page</td>
</tr>
<tr>
<td>---------</td>
<td>------</td>
</tr>
<tr>
<td>Delimitations and Significance</td>
<td>65</td>
</tr>
<tr>
<td>Methods of Analysis of Data</td>
<td>68</td>
</tr>
<tr>
<td>5. LEVELS AND PATTERNS OF ACHIEVEMENT</td>
<td>71</td>
</tr>
<tr>
<td>Performance on the National Assessment Items</td>
<td>71</td>
</tr>
<tr>
<td>Achievement on Graph Reading, Variation, and Variation in Variation</td>
<td>75</td>
</tr>
<tr>
<td>Conclusions</td>
<td>82</td>
</tr>
<tr>
<td>6. THE CARTESIAN REPRESENTATION</td>
<td>84</td>
</tr>
<tr>
<td>Understanding the Cartesian Convention</td>
<td>85</td>
</tr>
<tr>
<td>Working with Scales</td>
<td>87</td>
</tr>
<tr>
<td>Obtaining Information from Graphs</td>
<td>93</td>
</tr>
<tr>
<td>Conclusions</td>
<td>96</td>
</tr>
<tr>
<td>7. VARIABLES AND FUNCTIONAL DEPENDENCE</td>
<td>98</td>
</tr>
<tr>
<td>The Notion of Variable</td>
<td>99</td>
</tr>
<tr>
<td>Properties of Variables</td>
<td>102</td>
</tr>
<tr>
<td>Identification of Variables</td>
<td>104</td>
</tr>
<tr>
<td>The Notion of Dependence</td>
<td>106</td>
</tr>
<tr>
<td>Conclusions</td>
<td>109</td>
</tr>
<tr>
<td>8. VARIATION</td>
<td>111</td>
</tr>
<tr>
<td>Representing Variation</td>
<td>111</td>
</tr>
<tr>
<td>Interpreting Variation</td>
<td>119</td>
</tr>
<tr>
<td>Use of the Idea of Variation</td>
<td>124</td>
</tr>
<tr>
<td>Conclusions</td>
<td>128</td>
</tr>
<tr>
<td>9. VARIATION IN VARIATION</td>
<td>130</td>
</tr>
<tr>
<td>Concept of Rate of Change</td>
<td>131</td>
</tr>
<tr>
<td>CHAPTER</td>
<td>Page</td>
</tr>
<tr>
<td>-----------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>Comparison of Rates of Change</td>
<td>134</td>
</tr>
<tr>
<td>Perceiving and Using Regularities in Variation</td>
<td>141</td>
</tr>
<tr>
<td>Distinguishing and Representing Forms of Variation</td>
<td>145</td>
</tr>
<tr>
<td>Conclusions</td>
<td>154</td>
</tr>
<tr>
<td><strong>10. IMPLICATIONS.</strong></td>
<td>156</td>
</tr>
<tr>
<td>Summary of Conclusions</td>
<td>157</td>
</tr>
<tr>
<td>Implications for Further Research</td>
<td>161</td>
</tr>
<tr>
<td>Recommendations for Secondary School Mathematics Teaching</td>
<td>163</td>
</tr>
<tr>
<td><strong>REFERENCES.</strong></td>
<td>168</td>
</tr>
<tr>
<td><strong>APPENDIX</strong></td>
<td></td>
</tr>
<tr>
<td>A. Test of Graph Reading and Interpretation</td>
<td>183</td>
</tr>
<tr>
<td>B. Handouts for Graphing Tasks.</td>
<td>198</td>
</tr>
<tr>
<td>C. Coding Categories for Qualitative Data</td>
<td>211</td>
</tr>
</tbody>
</table>
LIST OF TABLES

<table>
<thead>
<tr>
<th>TABLE</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Alpha Coefficients and Standard Errors of the Mean for Three Subtests</td>
<td>57</td>
</tr>
<tr>
<td>2. Graphing Tasks and Subtest Scores by Student</td>
<td>64</td>
</tr>
<tr>
<td>3. Facilities on National Assessment Items for the NAEP samples and the Groups in the Sample</td>
<td>73</td>
</tr>
<tr>
<td>4. Raw Score Means and Standard Deviations (in Parenthesis) on the Subtests by Each Group</td>
<td>76</td>
</tr>
<tr>
<td>5. Summary of Analysis of Variance for Achievement in Graph Comprehension</td>
<td>77</td>
</tr>
<tr>
<td>6. Scaled Score Means and Significance Tests for the Group Differences in Subtest Means</td>
<td>79</td>
</tr>
<tr>
<td>7. Facility Values for Graph Reading Subtest</td>
<td>86</td>
</tr>
<tr>
<td>8. Distribution of the Number of Correct Responses to Items 1, 2, and 3</td>
<td>88</td>
</tr>
<tr>
<td>9. Types of Difficulty in Working with Scales</td>
<td>90</td>
</tr>
<tr>
<td>10. Inaccuracies in Reading Ready-Made graphs</td>
<td>94</td>
</tr>
<tr>
<td>11. Names and Classifications of Variables</td>
<td>101</td>
</tr>
<tr>
<td>12. Facility Values for Graph Construction Item</td>
<td>105</td>
</tr>
<tr>
<td>13. Representing Variation and Linearity</td>
<td>114</td>
</tr>
<tr>
<td>14. Facility Values for Variation Subtest</td>
<td>120</td>
</tr>
<tr>
<td>15. Strategies in Interpolation</td>
<td>126</td>
</tr>
<tr>
<td>16. Facility Values for Variation in Variation Subtest</td>
<td>136</td>
</tr>
<tr>
<td>Table</td>
<td>Title</td>
</tr>
<tr>
<td>---------</td>
<td>----------------------------------------------------</td>
</tr>
<tr>
<td>17</td>
<td>Strategies for Comparison of Rates of Change.</td>
</tr>
<tr>
<td>18</td>
<td>Representation of Nonlinear Variation</td>
</tr>
<tr>
<td>19</td>
<td>Representation of Smooth Variation and Discontinuity</td>
</tr>
</tbody>
</table>
CHAPTER 1

FUNCTIONAL REASONING AND GRAPH COMPREHENSION IN SCHOOL MATHEMATICS

The research problem of this study was twofold: (a) to identify high school students' and preservice teachers' levels and patterns of achievement in reading, constructing, and interpreting Cartesian graphs and in analyzing the underlying functional relationships; and (b) to describe some of their reasoning strategies and conceptual difficulties.

This chapter outlines the educational significance of the investigation. It surveys the importance ascribed to functional reasoning and graph comprehension by mathematics educators and contrasts them with students' attainments.

Functional Reasoning and Graph Comprehension as Educational Objectives

The concept of function was developed to express in mathematical terms the idea of dependence (Caraça, 1958; Shuard & Neil, 1977) and quickly became a fundamental concept of mathematics. Infinitesimal analysis, one of the
most successful branches of mathematics, is based on this concept, which is also an instrument of prime importance in the natural, social, and behavioral sciences as well as in engineering. The increasingly widespread presence of the computer in so many domains of modern life and its availability in schools has strengthened the importance of this concept (Fey, 1984).

Early in the 1900s Klein (1908/1945) argued that the notion of a numerical function "should permeate... the entire mathematical instruction in the higher schools" (p. 205). And indeed, the concept of numerical function is in many countries at the heart of the high school mathematics program. In the United States it plays an important role in the traditional Algebra I, Algebra II, and Trigonometry courses, and recent proposals for curriculum change have recommended that the function concept form the basis of the high school curriculum (College Board, 1983; Fey, 1984).

Cartesian graphs are closely related to numerical functions. In fact, a numerical function has one form of representation in a Cartesian graph. Many mathematical properties of functions are so well described by their graphs that the study of a function--determining intervals where it is increasing, decreasing, or constant; determining zeros, discontinuities, and asymptotes; searching for symmetry and periodic properties; etc.--is usually undertaken with reference to a sketch of its geometric representation.
Cartesian graphs are useful in communicating information but are also powerful analytical tools that can be employed to study complex phenomena (Elkins & Wockenfuss, 1972; French, 1982; Janvier, 1978; Shilov, 1978). From them it is possible to derive qualitative information and to draw inferences about relationships among variables as well as specific quantitative data.

The wide use of graphs in mathematics instruction may also play a positive role in the development and reinforcement of many mathematical concepts (Christopher, 1982; Hamley, 1934; Sullivan & O'Neil, 1980). A mastery of the "language of graphs" is an important basis for the study of further mathematics.


Applying the concept of function in more than trivial ways and reading, constructing, and interpreting graphs should be regular activities in middle and high school mathematics instruction. They should relate and support
the development of other mathematical topics. But these activities rarely have been coherently addressed in school programs (Bell & Janvier, 1981; Janvier, 1978; McKenzie & Padilla, 1983; Saltinski, 1983; Slaughter, 1983).

Students' Difficulties in Functional Reasoning and Graph Comprehension

Students are usually able to do some elementary graphing tasks well, like finding maximum or minimum values or plotting points given their Cartesian coordinates (Bamberger, 1942; Bestgen, 1980; Carpenter, Lindquist, Mathews, & Silver, 1983; Lindquist, Carpenter, Silver, & Mathews, 1983; McKenzie, 1984; Riggs, 1967; Shaw, Padilla, & McKenzie, 1983; Strickland, 1938; Thorp, 1933). Students can also easily develop a reasonable proficiency in handling routine exercises involving numerical functions (Dreyfus & Eisenberg, 1982; Goldberg, 1962/1975; Smith, 1973; Thomas, 1971, 1975). These tasks have a procedural nature in that precise directions can be established to execute them correctly.

But students' performance on more complex tasks tends to be very unsatisfactory. Students have great difficulty in relating the Cartesian representation of simple lines and curves to the corresponding algebraic equations as well as in interpreting the qualitative information implicit in the graphs (Barr, 1980, 1981; Bestgen, 1980; Carpenter,
Corbitt, Kepner, Lindquist, & Reyes, 1981; Kerslake, 1977, 1981; Shaw, Padilla, & McKenzie, 1983; Wagner, Rachlin, & Jensen, 1984) or in using all the relevant information given to discuss the features of functional relationships (Janvier, 1978; Karplus, 1979; Markovits, Eylon, & Bruckheimer, 1983). Many students seem not to have developed an adequate understanding of what Cartesian graphs are, how they represent relationships between variables, and what information can be derived from them. Even at the college level, students sometimes experience difficulties in graph interpretation (Konshak & Monk, 1976; Vernon, 1950).

This situation is hardly surprising if one recognizes that the interpretation of graphs and the application of the concept of function to real situations and natural phenomena are usually neglected in mathematics classes. Like most modeling activities, these complex tasks do not admit simple well-defined procedural methods for their solution. Instead, to be worked out successfully, they require a global understanding of the underlying situation and the use of a variety of intuitions and background knowledge.

Students' learning experiences in functional reasoning and in interpreting graphs constitute an important area of research in mathematics education. The positive results obtained in teaching elementary and middle school children the fundamental principles involved in reading and constructing simple graphs and in handling instrumentally
numerical functions sharply contrast with the profound difficulties experienced by high school and college students in solving more complex problems involving graph interpretation and functional reasoning. Mathematics educators need to identify what may constitute successful learning experiences to deal with the interpretation of complex graphs. This research was carried out in the belief that further investigations were needed on students' abilities, conceptions, processes, and difficulties in this topic.
This chapter presents a view of mathematical thinking processes that was developed to constitute the general theoretical background of the study and reviews and discusses closely related theories about thinking, understanding, imagery, and intuition. The chapter also describes the framework developed to study functional reasoning and graph comprehension.

Mathematical Thinking Processes

Concepts and conceptual structures may be processed through step-by-step mental operations or in a more integrated manner. Mental operations may be of various types. This section describes several aspects of conceptual operations and asserts the existence of three fundamental kinds of reasoning processes in mathematics.

Operations and conceptual structures. Some conceptual operations are analogous to mathematical operations, whereas others have no direct counterpart in mathematics. For example, relational operations include establishing
conceptual structures are strong and using them may be just a routine process.

**Mathematical reasoning processes.** It is possible to distinguish three fundamental kinds of reasoning in mathematics. They may coexist and interrelate in many problems, but nevertheless their role is almost always identifiable. Each kind of reasoning fulfills a different purpose and may be most appropriate for a specific kind of task. They are:

1. Logical-deductive reasoning. It is used in deduction and consists mainly of arguments of the "if-then" form; it depends on propositions previously accepted and on the exploration of consequences of definitions. Its steps are well defined, and the concepts manipulated are usually fairly abstract. Logical-deductive reasoning is fundamental in the process of validating and organizing mathematical knowledge and was the main focus of Piaget's (1972; Inhelder & Piaget, 1955/1958) investigations.

2. Algorithmic reasoning. It consists of sequences of well-defined steps and is used to solve a whole class of similar problems. In each step a single, well-defined operation is made, usually manipulating fairly concrete concepts. Algorithmic reasoning is very powerful in solving routine problems, although the number of necessary steps may be very large. This form of reasoning has been brought to prominence by the development of computer science (Knuth, 1974).
global-visual symbols range from simple pictures of objects to complex representations in maps and diagrams. Their apprehension and interpretation is fundamentally a holistic process. Analytical-verbal symbols include the spoken and written words of common speech and the most common algebraic symbols. This kind of symbolism is processed in a sequential way (Paivio, 1971).

These two kinds of symbols correspond to two modes of thinking. The operations most used in global-visual reasoning are association, substitution, and geometrical operations. Analytical-verbal thinking makes wider use of logical, relational, and arithmetical operations.

Graph reading and graph interpretation share common features as visually-based tasks. But for successful handling they demand different reasoning processes. Graph reading can be processed in a procedural manner, that is, it is possible to set forth a set of directions that apply to whole classes of problems. Graph interpretation has a more open-ended nature. It requires the creative involvement of a large number of conceptual structures concerning the situations represented. Graph interpretation has to be processed with the intervention of intuitive reasoning.

The procedural perspective of mathematics held by so many students may partially explain why they tend to do relatively well in the simpler graph reading tasks but find
preoperational, concrete operations, and formal operations. The factors explaining the development are maturation, experience, social transmission, and equilibration.

Piaget distinguished two kinds of experience. Physical experience consists of action upon objects and leads by abstraction to inferences about the objects. Logical-mathematical experience leads to the drawing of knowledge not from the objects themselves but from actions performed with the objects. This knowledge concerns properties of actions on the objects and not properties of the objects themselves. For Piaget (1964a) mathematical deduction begins with logical-mathematical experience and its suitable combination with symbolization.

Piaget (1964b) claimed that there is a close correspondence between the mathematical structures identified by the Bourbaki school and the psychological structures of human intelligence. Piaget studied the psychological structures by formalizing the operations used by children and concluded that they were analogous to but much simpler than the mathematical structures.

The notion of operation played an important role in this study, but it was used in a broader sense than that of Piaget. Some psychological operations were regarded as analogues of mathematical operations, but others with no clear mathematical counterpart were considered as well.
to self-perpetuation becomes an obstacle to adaptation. This resistance may be very strong, especially where the scheme plays a vital role for the individual.

Mathematical concepts neither exist nor can be learned in isolation from each other. The notion of conceptual structure is therefore of great importance in modeling mathematical learning. It played a major role in this study.

**Imagery**

Skemp (1971) referred to two kinds of thinking, based on two different systems of symbolic representation. Discursive-verbal thinking focuses attention on only one part of a scheme at a time. This kind of thinking is especially suited for analytical tasks. Global-visual thinking is holistic and concerns the ways in which parts relate to each other and to the whole. Global-visual thinking is especially suited for synthetic enterprises. Skemp suggested a complementary role for these two forms of language systems, indicating, however, that the socialized character of our knowledge and our difficulty in externalizing images may have caused an imbalance in the development of our ability to use these two modes of thinking.

In studying the role of visual imagery Piaget and Inhelder (1971) concluded that images were subordinated to operations. But they indicated that images constituted a
The distinction between discursive-verbal and global-visual thinking played an important role in this study as it assessed how successful students were in using both kinds of reasoning in graph-related tasks. Images were regarded as representing concepts, on equal grounds with other symbols, and so also as subject to manipulation through operations.

Intuition

According to Fischbein (1979, 1982) our intelligence works by interrelating two modes of operating: the intuitive and the logical-analytical. At the intuitive level we think in a global, partly conscious and partly unconscious fashion. We do not attempt to make explicit all the concepts and schemes that we are using. Instead, in solving a problem, for example, we put all our capabilities to work, hoping that the desired connections of ideas will result. Intuition is a global and compact form of knowledge with a unique quality of self-evidence that results from our apprehension of the structure of a given situation.

In Fischbein's view the essential feature of logical-analytical thinking is its explicitness. Logical-analytical thinking proceeds at the conscious level using refined symbolic media and can easily be communicated. It has, however, two essential shortcomings: it proceeds slowly, and it is not oriented towards action. Also, logical-analytical thinking develops relatively late in human beings.
be achieved by combining two components, the logical form of necessity that is characteristic of mathematical proof and the internal structural form of necessity that is characteristic of intuitive acceptance.

In school the process of refining and correcting one's intuitions is usually disregarded or driven in a haphazard manner. Fischbein (1973) pointed out that intuitions are not developed by blind drill and practice nor by explanations or short learning exercises. He claimed that they "can be elaborated only in the frame of practical situations as a result of the personal involvement of the learner in solving genuine problems raised by these practical situations" (1982, p. 12). An important element of this process, according to Fischbein, is the reassessment by students of the elements of their primitive experience in the light of the framework provided by a rigorous mathematical interpretation.

Fischbein asserted that teaching may relate to the students' initial intuitions in one of three ways: (a) it may be in accordance with them, in which case there is immediate acceptance and reinforcement; (b) it may be in sharp opposition to the students' intuitions, in which case it will be rejected or ignored; (c) it may be in some sense "neutral," neither opposing nor favoring previous intuitions. In this last case the intuitions have to be developed and transformed into acquired knowledge or else they will remain precarious and ineffective. The case of total
a well-defined way on the value taken by another variable. Graph comprehension refers to a wide spectrum of processes, including constructing graphs and obtaining quantitative and qualitative information from graphs.

The framework through which students' ability to perform tasks in functional reasoning and graph comprehension was analysed has three aspects: (a) graph reading, (b) graph construction, and (c) graph interpretation. Each of these aspects is discussed in detail below.

**Graph reading.** Graph reading corresponds essentially to the understanding and use of the basic conventions and principles of the Cartesian representation. The focus is on the coordinates of single points. Graph reading involves for the most part procedural tasks, that is, tasks in which students can be instructed to follow a sequence of steps that, if carried out correctly, leads to the desired goal. The graphs may be abstract or they may represent real-life situations. Specifically, graph reading refers to tasks that may involve identifying the coordinates of points, indicating the value of a function corresponding to a given value of the abscissa, indicating the abscissa corresponding to a given value of a function, identifying the coordinates of points where the graphs of two functions intersect, and comparing the values of a function for distinct given abscissas.

**Graph construction.** Graph construction refers to a wide range of tasks such as identifying the variables
problems. It may require an extensive use of intuitive reasoning. It constitutes the most complex and for the students the most difficult aspect of graph comprehension. Interpretation of variation in variation involves notions related to rates of change, such as fast and slow change, linear and nonlinear change, continuous and discontinuous change, smooth and nonsmooth change. Tasks addressing this aspect of graph interpretation may include comparing rates of change, identifying graphs representing complex patterns of variation, describing features of a situation represented by nonlinear graphs, and using curvilinear interpolation and extrapolation.

These three aspects of graph comprehension--graph reading, graph construction, and graph interpretation--are certainly interrelated as components of graphing ability, but they are viewed in this framework as relatively independent processes. Graph reading may be carried out with little reference to the situational context, whereas graph construction requires the student to work from a situation to a graph and graph interpretation requires the student to relate features of the graph to features of the situation. The interpretation of variation is ordinarily processed using concepts closely related to the situation, whereas the interpretation of variation in variation is thought of as being processed using more abstract and elaborated concepts.
CHAPTER 3

REVIEW OF RESEARCH

Several studies concerning the concept of function have been undertaken since the mid-1960s, and studies dealing with graph comprehension have been conducted since the beginning of the 1920s. These studies covered a wide range of grade levels from kindergarten through college. Reviews focusing on the relative effectiveness of different kinds of graphical materials were provided by Malter (1952) and Macdonald-Ross (1977). Using the framework of this study, the present chapter reviews research related to functional reasoning and graph comprehension.

Research on Functional Reasoning

The concept of function started attracting attention of researchers in mathematics education in the late 1960s and early 1970s. A dominant research paradigm at that time was the search for stages in the process of forming a concept (Andersen, 1971; Lovell, 1971; Orton, 1970; Piaget, Grize, Szeminska, & Bang, 1968; Thomas 1971, 1975).
were greater than had been commonly thought by mathematics educators (Herscovics, 1982).

More recently, another strand of research has developed that is aimed at the identification of students' abilities and difficulties in dealing with functions and at the study of students' conceptions and psychological images of functions.

The following pages review results, findings, and implications that can be drawn from the studies regarding students' ideas of variables and functional dependence, variation, and variation in variation.

**Variables and Functional Dependence**

In simple real-life contexts involving natural numbers and proportionality, elementary school students do not seem to have difficulty with the notion of the uniqueness of images (Ricco, 1982). However, it appears that older students who are taught about functions in a formal way, just emphasizing abstract correspondences and disregarding the stimulating role of the underlying contexts, may fail to understand the idea of functional dependence. For example, Thomas (1971) was puzzled by his observation that students in the Secondary School Mathematics Curriculum Improvement Study were able to handle instrumentally routine exercises but could not distinguish very simple instances of functional and nonfunctional relationships.
of the meaning of the term "dependence" in everyday language.

Indeed, Bang (Piaget, Grize, Szeminska, & Bang, 1968) suggested that it was easier for young children to grasp functional dependencies of a qualitative character when these dependencies related to situations involving a causal consequence. But from this primitive idea of dependence, children still seem to have a long way to go until they establish well-defined rules of covariation between given variables and develop the ability to use these rules to predict the results under new experimental conditions.

The notion of variable. Closely related to the difficulties that sometimes exist with the notion of dependence are difficulties with the notion of variable. Many young students often seem to lack a workable concept of variable, not going beyond the notion of "unknown" (Kuchemann, 1978, 1981; Marnyanskii, 1965/1975). Older students often view a variable—as did 18th-century mathematicians—as something running through an unlimited number of values rather than as a general representative of a given set (Marnyanskii, 1965/1975). In word problems, college students were found by Rosnick (1982) to frequently fail to assign variables to well-defined referents, letting them instead represent an undifferentiated conglomerate of meanings.

A quite interesting feature of students' thinking is that they do not seem to relate in a spontaneous way the ideas of variable and set. Marnyanskii (1965/1975) found
students handled the situation in a pointwise manner, referring to the points with integer abscissas.

It would appear that the notion of functional dependence is essentially constructed on the basis of relationships imbedded in situational contexts. Many of these are commonly modeled using continuous variables. Therefore, the nature of the students' conceptual difficulties in handling such variables deserves closer attention.

**Conceptual structure of functional dependence.** If some students initially experience a number of difficulties with the idea of dependence, many of them later seem to adopt the notion of a well-regulated dependence as the essence of the concept of function. Research conducted by Wagner (1977, 1981) showed that many students were easily misled into error by the symbols used to represent the variables but resorted frequently to a strategy of "searching-for-the-rule" as a way of coping with unfamiliar problems. Vinner (1983) showed that even when taught an abstract definition of a function, 10th- and 11th-grade students worked hard to find the "algebraic rules" that would supposedly legitimize some correspondences as "true functions." For some students a function should correspond to a single rule, but others were willing to accept the idea that a function could be defined by different rules in distinct parts of its domain.

Even in mathematically more mature subjects the concept image may be rather distinct from the concept
student did not understand that the values of a function "could vary only with different argument values [and that] the values of a function vary depending on the change in the values of the argument" (p. 150-151). Janvier (1978) also reported that many students had difficulties with the concepts of change, growth, and increase.

Problems in adequately understanding the nature of variables may also interfere with the notion of variation. For example, Janvier (1981) reported that many students indicated as the point where a function started increasing not the value corresponding to the minimum but the next whole value. Complex forms of variation may also disturb the understanding of the relevant variables. Lochhead (1980) noted the tendency to confuse variables when a rate of change was involved or when there was interference from pictorial representations. Trowbridge and McDermott (1980, 1981) showed that in the interpretation of physical experiments students easily confuse position and velocity or velocity and acceleration.

Achievement results. Kuchemann found test items requiring an understanding of how the values of a variable expression change to be rather difficult, even for 15-year-olds (less than 10% correct responses), and exceedingly difficult for younger students.

Sometimes students demonstrate some intuitive understanding of the variation represented in a certain question but cannot express it in abstract terms. For example, in a
basis for functional reasoning is laid by 12 or 13 years of age. However, data on students' achievement also suggest that this potential may not be achieved in formal tasks that are not experimentally based. Bang's research also suggests that conceptual problems in other areas of mathematics may hinder functional reasoning. For example, it seems likely that a working familiarity with the rational number system may be an important prerequisite to handling successfully many functional situations.

**Linearity.** Students seem to develop and make wide use of a conceptual structure in which all functional relationships have a linear form, even when presented with other kinds of functions during regular instruction. It would appear that linear functions, besides being the most widely used functions in mathematics and science courses, contain an inherent simplicity that makes them very appealing to the human mind.

Markovits, Bylon, & Bruckheimer (1983) asked ninth graders to give examples of functions satisfying certain constraints both in algebraic and graphical representations and in pure mathematical and scientific contexts. Most of the responses were restricted to linear (or piecewise linear) functions, and many students did not recognize that nonlinear functions could also be solutions to the same problems. Markovits et al. concluded that "students have a mostly linear image of functions, which is not influenced, either by the type of representation, or by the kind and by
reasoning and also has a striking historical analog. Linear variation and proportions pervaded most ancient and medieval mathematical thought and were the embryonic form of functional thinking that were finally extended to a general concept of function by 17th- and 18th-century mathematicians.

**Intuition and geometric representations.** Markovits, Eylon and Bruckheimer (1983) indicated that some students, when asked to give their answers in algebraic terms, translated the problems into a graphical form and used geometrical explanations in their answers. This response seems to suggest that on some complex tasks the geometric intuition, although still not fully developed, may already be playing an important role in the development of the reasoning of some students.

**Conceptual structure of variation.** As soon as students start grasping the idea of variation, they have a common tendency to regard as functions only things that vary (Markovits, Eylon, & Bruckheimer, 1983; Marnyanskii, 1965/1975), in a perspective reminiscent of the Leibnitzian view of constants and variables as mutually exclusive entities. However, most students appear to not have difficulty in adopting later a more general view (accepting constant functions) as a natural generalization of the patterns of linear variation.

Goldberg's (1962/1975) research seems to support the idea that the study of linear variation in an algebraic
another thing to understand regularities that constitute patterns of rate of change.

Rate of change. Trowbridge and McDermott (1980) reported that it was common for students untrained in physics to view speed as an imprecise association between distance and time and not as a ratio between these two quantities. When the concepts involved were more abstract, as in thinking of acceleration as the instantaneous rate of change of velocity with respect to time, Trowbridge and McDermott (1981) indicated that the conceptual difficulties experienced by the students appeared to be very persistent.

Lochhead (1980) pointed out the difficulties that college students have in using an algebraic concept of rate of change to solve word problems. He noted that students commonly confuse the amount of a quantity with its rate of change. The research of Trowbridge and McDermott (1980) also illustrates the very common tendency to confound amount and rate in the context of simple kinematic experiments.

Janvier (1980) asked students for the greatest increase of a function represented in a Cartesian graph. The students very often did not search for the points (or intervals) with greatest slope but interpreted "largest increase" as "being large" or as "starting to increase." Janvier ascribed these incorrect responses to attraction by high values (that is, focusing on the upper part of the curve) or attraction by low values (focusing on the lower
again and again strikingly supported by the research on functions, which has been undertaken by researchers using different methodologies and working within different theoretical orientations.

The investigations carried out by Piaget and his associates (1968) support the idea that the intuitive basis for dealing with functions involving direct and inverse proportionality may be developed in 12-year-olds provided that they are given adequate physical experiences upon which to build the relevant concepts regarding the variables and the relationships involved. The failure of many school programs to develop an understanding of such notions at this age or even at a later age may be found in the conventional ways that schools tend to operate—emphasizing abstract representations, routine computations, and teacher talk, instead of promoting active involvement of the students in the construction, development, and application of mathematical ideas.

Goldberg's (1962/1975) teaching experiment seems to suggest that students' understanding of the notion of functional dependence can be stimulated by a mathematically rich foundation that relates to the conceptual structures the students have already acquired and that promotes the development of their intuitions.
Graphs do not seem to be a difficult subject to teach effectively at the elementary and middle school levels (Bamberger, 1942; Herrmann, 1976; Johnson, 1971; McKenzie, 1983; Riggs, 1976; Strickland, 1983; Thorp, 1933). However, there is a lack of consensus about the grades in which to introduce different kinds of graphs. The studies that have dealt with relative instructional effectiveness and optimal grade placement (Goetsh, 1936; Mathews, 1926; Thomas, 1933; Washburne, 1927a, 1927b; Wrightstone, 1936) have reported conflicting results for the most part (Malter, 1952). Reviewing these studies, Weintraub (1967) suggested a "developmental sequence" that introduced--successively--pictographs, circle graphs, bar graphs, and finally line graphs, whereas Appel (1973) recommended that all types of graphs should be introduced as early as kindergarten.

Other studies compared different forms of presentation of data to adult audiences. Questions involving quantitative aspects produced lower performance in general when presented by means of line graphs than when presented by means of tables or other types of graphs (Carter, 1947; Culbertson & Powers, 1959; Vernon, 1946). These studies indicated that if specific values are to be read by the students from line graphs, dealing with scales and reading accuracy may be serious problems. Although the presence of coordinate rulings may be a factor in increasing accuracy, denser or wider rulings do not seem to make much difference (Carter, 1947).
Graph Interpretation

Line graphs are the form of representation of data usually selected when the purpose is to study complex trends. Indeed, Shutz (1961) found that performance in situations involving trends was better with line graphs than with bar graphs. However, the interpretation of graphs has been found to be generally very difficult for most students. Moderate to very low performance has consistently been found on questions involving aspects such as choosing the appropriate graph to represent a given situation, describing relationships between variables, interrelating the information from several graphs, and interpolating and extrapolating information (Carpenter et al., 1981; Janvier, 1978; Shaw, Padilla, & MacKenzie, 1983).

Janvier (1978) described graph interpretation as a process of progressive integration of the various pieces of information conveyed by the graph with the students' background knowledge of the situation. Interpretation, then, would consist of the association of global features of the graph with the facts of the situation. He claimed that verbal formulations played a critical role in the processes of translation between the situations and the graph, even when the student worked from a diagram to the graph.

A major problem concerning the interpretation of graphs seems to be posed by the complexity of the situations involved, including the nature of the data presented
reported that in questions on distance/time graphs, which in England normally receive some attention in most schools, a considerable number of students had difficulties when the graph could be visually misleading. Janvier (1978, 1979) indicated that sometimes students mix together symbolic and pictorial interpretations of the same graph.

The interpretation of graphs to derive specific quantitative information also seems to pose serious problems. Carter (1947) noted that Air Force pilots and college students were very slow and inaccurate in interpolation problems. The rate of change seems to pose very hard conceptual difficulties for many students, who easily confuse intervals of greatest increase and intervals of greatest value (Janvier, 1978; Bell & Janvier, 1981). Comparisons of rates of change also appear to present problems to students, being more difficult than reading or comparing simple values (Herrmann, 1976; Janvier, 1978; Price, Martuza, & Crouse, 1974).

Many students seem to consider a graph as a mere source of pointwise information. This has been found for middle school students (Janvier, 1978) and for college students (Konshak & Monk, 1976). Besides, students sometimes seem to have unexpected ideas and misconceptions about graphs. For example, Karplus (1979) reported that some students considered that a graph representing the behavior of living organisms could not be composed of straight lines. Other students thought that straight lines were
misleading graphs. Johnson (1971), on the other hand, reported correlations between graph reading and spatial ability from .43 to .45 and between graph reading and perceptual speed from .32 to .54.

It appears that the proportion of the variance in students' graph comprehension that each of the previous variables accounts for is significant but not very large.

Conclusion

Previous research seems to indicate that the fundamental principles involved in reading Cartesian graphs can indeed be learned by elementary and middle school children and also that a deficient understanding of the geometrical representation of rational and real numbers may induce difficulties in handling continuous variables. These difficulties become particularly evident in graph construction tasks.

The interpretation of graphs seems to be a more complex and difficult process than just reading graphs. To the complexity of representing processes of change is added the complexity of the situations involved. Students appear to use graphs mainly as a source of pointwise information, although sometimes they view them in a pictorial manner. As a result, they tend to do very poorly in deriving both qualitative and quantitative interpretive information from graphs.
"intrinsically accurate," and so the best interpolation strategy to use would always be linear interpolation.

**Relationships Between Graph Comprehension and Cognitive or Achievement Variables**

Some investigators studied the correlations between aspects of graph comprehension and cognitive variables such as general intelligence, developmental stage, or achievement variables such as arithmetic and reading ability.

Using IQ tests, Culbertson and Powers (1959), Herrmann (1976), Johnson (1971), and Riggs (1976) reported correlations between graph reading and general intelligence ranging from .43 to .62. With instruments to assess level of cognitive development, McKenzie (1984) and Padilla, Okey, and Dillashaw (1983) found correlations between graphing ability and developmental stage ranging from .36 to .77.

Arithmetical or mathematical ability is also correlated with graph comprehension. Culbertson and Powers (1959), Curcio (1982), and Herrmann (1976) found correlations ranging from .54 to .73. Reading ability was found by Curcio (1982), Herrmann (1976), and Johnson (1971) to correlate between .60 and .70 with graph comprehension.

The relation of graph comprehension to spatial abilities is not clear. Kerslake (1977) claimed that students who are strong visualizers have more difficulty than other students in interpreting some of the most potentially
and the concepts and technical terms used (Janvier, 1978; Konshak & Monk, 1976; Vernon, 1953). A student may be able to answer specific questions about a graph, but that fact does not guarantee a global understanding of the situation (Vernon, 1950). Some familiarity with features of the underlying situation may be a decisive factor in helping students make correct interpretations (Janvier, 1978). Janvier (1978; Bell & Janvier, 1980) also noted the perturbing influence of situational distractors in the process of graph interpretation. That is, previous experience with the situation can interfere with the interpretation of abstract features of a graph.

A number of other investigators have identified difficulties in setting up a correspondence between situations and graphs. Woodward and Byrd (1984) related how, in constructing stories to explain graphs, students often disregarded important aspects of the graphs. Lochhead (1983) indicated the difficulties of college students in identifying velocity/time and acceleration/time graphs describing simple motions. Karplus (1979) reported that the responses of many students to his tasks did not reflect reasoning about the situation that had yielded the data. Rather, most students who attempted a response had followed some procedure that they supposedly recalled from their classes.

Students often attempt to use a graph in a very direct manner as a pictorial representation of a situation (Janvier, 1978; Konshak & Monk, 1976). Kerslake (1977, 1981)
Graph Construction

Wavering (1981, 1983) investigated the thought processes that middle and high school students used in graphing data. Wavering sought to validate a developmental sequence for graphing, involving (a) the ordering of data, (b) the scaling of the axes, and (c) the recognition of a relationship from the graphed data. Some students were unable to identify a relation underlying given data just because of difficulties in plotting the points, which in some tasks was quite a demanding problem because of a large and unusual range of data.

An unquestionable finding of Wavering's investigations is that graph construction may pose quite serious problems for high school students. Shaw, Padilla, and McKenzie (1983) also found that choosing appropriate scales was one of the most difficult aspects of a test of graph competence. Coward (1981) reported that many junior college students, although able to construct acceptable scales to represent ordered pairs, considered this an awkward task.

The implications of the use of graph paper in the construction of graphs was addressed in Wavering's (1983) study. Graph paper seemed to help middle school students order data and scale axes but had no effect on the performance of high school students.
Previous research has investigated several aspects of graph reading, construction, and interpretation. Relationships between performance on graph comprehension and cognitive abilities or school achievement have also been studied from a correlational viewpoint.

Graph Reading

Working with simple graphs seems to be a suitable activity for the elementary and middle school levels. At some stage in their cognitive development, young children begin to understand the principles of Cartesian representation (Piaget, Inhelder, & Szeminska, 1960).

Reading graphs and plotting points have consistently been identified as the aspects of graph comprehension with the highest levels of success (Bestgen, 1980; Carpenter et al., 1981; Coward, 1981; Kerslake, 1977, 1981; Price, Martuza, & Crouse, 1974; Shaw, Padilla, & McKenzie, 1983). However, difficulties in dealing with scales seem to trouble in many high school students and adults. These difficulties are probably related to a deficient understanding of the systems of rational and real numbers and their representation on the number line and appear not to be susceptible to easy remediation (Sullivan, 1982; Vergnaud & Errecalde, 1980).
part of the curve). Furthermore, Janvier claimed that these attractions can be strong or weak depending on whether the attention is directed only at the very extremes or comprises an interval from the point of largest increase to the extremes.

**Smoothness.** The notion of smoothness is associated with the mathematical ideas of continuity and differentiability. This notion and the idea that natural phenomena are often described by smooth functions seem not to be grasped by many students, even those who have studied functions in previous mathematics courses (Karplus, 1979).

However, when the notion of smoothness is grasped, it tends to play an overriding role. If a graph does not look "sufficiently reasonable," many students think that it cannot represent a function (Vinner, 1983). This resistance may be regarded as similar to that of many mathematicians in the past to the idea of discontinuous and nondifferentiable functions as legitimate mathematical entities and suggests that it is another natural step that students need to take in order to build a more general and abstract concept of function.

**Conclusion**

The idea that the difficulties that students have to overcome are similar to the difficulties that the great mathematicians experienced in the past (Kline, 1970) is
setting may be appropriate for eighth graders, despite the difficulties that it presents. Janvier's (1978) study suggests that linear variation represented in tables and graphs can also be understood by students at about the same grade level and can be meaningfully integrated into a single conceptual structure for linear variation. But an open question is how resistant the students' fixation on the pattern of linearity is and what experiences may lead them to construct wider and more flexible conceptual structures.

**Variation in Variation**

Variation in variation, and in particular the rate of change, seems to pose difficult conceptual problems for many students. This difficulty is probably related to the fact that students appear to develop and make frequent use of a conceptual structure that refers to a constant form of variation, namely linear variation.

Bang (Piaget, Grize, Szeminska, & Bang, 1968) reported that in very simple situations some 12-year-olds (but not younger children) were sensitive to nonlinear variation. For example, some children realized that there was a smaller difference between sticks 10 and 25 cm long than between sticks 25 and 50 cm long. But it is one thing to grasp irregularities in the variation that challenge the conceptual structure of "universal proportionality," and
the number of constraints, and also not by the context" (p. 276). Karplus (1979) proposed to 6th through 12th graders a series of "functional puzzles" in which smooth nonlinear functions would be appropriate models to use. Most students showed either a complete absence of feeling for the problems and guessed the answers or used straightforward linear interpolations. Only a small minority (less than 8% of 6th graders to less than 30% of the 12th graders) used curvilinear interpolations.

The predominance of a conceptual structure involving direct proportional reasoning with a positive constant of proportionality may be traced in the reactions of some children participating in Bang's (Piaget, Grize, Szeminska, & Bang, 1968) investigations. In a problem implying a reciprocal compensation of two quantities (an increase in one implying a decrease in the other), young children were unable to grasp the relationship and seemed fixated on the idea that larger implies larger and smaller implies smaller.

Linearity appears to be, in some instances, more than just a geometric reasoning pattern. For example, Wenger and Brooks (1984) indicated that a high percentage of college precalculus students tend to use the additive linearity property \[ f(a+b) = f(a) + f(b) \] with nonlinear functions like the power function.

The role played by linear variation in students' conceptual structures is closely related to proportional
National Assessment problem (Carpenter et al., 1981) in which students were asked to find a missing number in a table according to a linear pattern, only 35% of the 13-year-olds answered the item correctly; only 3% were then able to express the relationship by an algebraic equation.

The intuitive basis for functional reasoning. The problems just described refer to functions as normally encountered in a school setting. Working with children in the context of experimentation with simple materials, Bang (Piaget, Grize, Szeminska, & Bang, 1968) concluded that before 7 or 8 years of age children do not relate the relevant variables and do not grasp the sense of the variation. Bang suggested a developmental sequence in which between 8 and 12 years of age children become aware of the qualitative and quantitative aspects involved in simple processes of variation. Ricco (1982) presented elementary school children simple proportionality tasks in real-life contexts. She noted that first the students were able to grasp qualitative aspects of the relationships involved (uniqueness of the images and monotonicity), next the students used additive strategies to find missing images, and finally they identified and used the constant multiplicative coefficient.

There is a gap between the results obtained by a patient interaction with children on a one-to-one basis and the outcomes of the process of regular schooling. The studies by Bang and Ricco seem to suggest that the intuitive
definition. Dreyfus and Vinner (1982) found that many mathematics teachers and college mathematics majors were able to give formal definitions of a function but indicated that "these concept definitions remained very often inactive at decision making moments when (sometimes wrong) concept images took over" (p. 17).

These findings support the view that for most students the construction of a conceptual structure of functional dependence is related much more to the examples that these students are used to working with than to the definitions that they are given. This conceptual structure, in which well-regulated functional rules play a major role, appears to be—as it was for the mathematicians who contributed to the historical development of the notion of function—a "natural" step in the process of building a more general concept where the emphasis is on the correspondences (or on the ordered pairs) and not on the rules.

**Variation**

Some students seem to be uneasy with the idea of variation. They have difficulty in distinguishing value and variation. In Goldberg's words: "No matter what and no matter where it occurs, students identify a plus sign with an increase and a minus sign with a decrease" (p. 148). Such students tend to focus on particular values and disregard the process of variation. Goldberg observed that often the
that many students did not have the idea that a mathematical variable was associated (even if only implicitly) with a certain domain. Marnyanskii also indicated that the concept of physical variable was sometimes improperly expanded to include notions that are of a subjective nature like success, mood, and attentiveness.

**Discrete and continuous variables.** Students seem to experience much more difficulty in dealing with continuous variables than with discrete variables. For example, Karplus (1979) reported that students tended spontaneously to view variables as dichotomous (with either "high" values or "low" values) rather than as continuous. Janvier (1982), summarizing his own research, also suggested that distinct conceptual structures are involved in handling discrete and continuous variables. Janvier claimed, furthermore, that practice with skills involving discrete variables may not be transferred to skills involving continuous variables.

Also, in interpreting Cartesian graphs representing continuous situations, this tendency to view variables as having discrete values can appear. Some students, for example, show a tendency to focus on single points, to subdivide lines and curves into discrete parts, and to jump to nearby points with integer coordinates. These tendencies were observed by Janvier (1981), who asked students to interpret graphs representing one or more functions imbedded in situational contexts. For example, when asked to determine when one variable was greater than another, many
Goldberg (1962/1975) also reported that after the study of a conventional instructional unit on functions many eighth graders actually did not grasp the idea of a functional correspondence. These students failed to realize that both the function and argument took numerical values and that to each value of the argument corresponded one and only one value of the function. The basic idea of functional dependence was not understood by these students.

Goldberg conducted a teaching experiment with two classes. He emphasized functional dependence in an algebraic setting through extensive practical work with simple numerical functions. He concluded that this approach was much more successful in developing the idea of functional dependence than an approach based on the more abstract notion of correspondence.

**Intuitive ideas about functional relationships.**

Marnyanskii (1965/1975) observed two tendencies in 8th- and 10th-grade students regarding the scope of the concept of functional relationship: either it was restricted to simple kinds of algebraic relationships or it was expanded to include situations where the variables were not well defined and the connection was somehow diffuse (for example, "a pupil's success is a function of his attentiveness," pp. 165-166). Also, the scope of the idea of dependence was sometimes restricted by the students to the idea of causal connection—which Marnyanskii attributed to the influence
The investigations undertaken by Piaget and his associates were primarily concerned with the epistemology of the concept of function, stressing the psychological significance of the idea of functional dependence. Two different kinds of functions were studied in the work of the Geneva school: (a) functions-in-process-of-formation ("fonctions constituentes"), that were described as preoperational schemes arising from the schemes of action that characterize the thinking of preschool children; and (b) constituted functions ("fonctions constituées"), that were regarded as based on the operational and causal structures that are found in mature thinking.

Subsequent researchers focused primarily, if not exclusively, on the framework of mappings. They emphasized abstract correspondences and dismissed numerical functions. Some stages in the development of the concept of function were hypothesized by Thomas (1971, 1975) and later were in part supported by Orton's (1970) and Andersen's (1971) investigations. But the principal finding that emerged from this research was that many students experienced great difficulties with a formal approach to the notion of function despite their general high ability. The results of the Second National Assessment of Mathematics concerning student achievement on items dealing with functions (Carpenter, Corbitt, Kepner, Lindquist, & Reyes, 1981) reinforced the idea that students' difficulties in handling this concept
appropriate to describe a given situation and capture its continuous, discrete, or uniform properties; plotting points; drawing, labeling, and scaling axes; and constructing, from empirical data, lines representing functional relationships. Some general guidelines can be given for constructing graphs but they have to be adapted or even modified to suit the specific features of each situation. Graph construction, therefore, involves procedural and nonprocedural aspects.

**Graph interpretation.** Graph interpretation includes the processes involved in interpretation of variation and in interpretation of variation in variation. *Interpretation of variation* refers to an understanding of representations of change and nonchange processes. The main focus is on the successive values of a function and concerns the notions of increase, decrease, constancy, maximum, and minimum. The interpretation of variation may involve tasks such as identifying graphs representing given situations, describing features of a given situation from the observation of graphs, using information from several parts of the graph to make statements about the implied situation, and using linear interpolation or extrapolation. The processes used in interpretation of variation have a more intuitive and less procedural nature than those used in graph reading.

In *interpretation of variation in variation* the focus is on the law of variation and concerns the understanding of its properties and their application to practical
opposition is even more complex, requiring the development of new, parallel intuitions that contradict the former ones.

The importance of intuitive thinking has been addressed in the psychological literature. Bruner (1960, 1966) stressed the superiority of intuitive over formal understanding and indicated its role in creative endeavors. Piaget (1973) attributed an operational character to mathematical intuition and indicated that in the learning process intuition should precede axiomatization. Also, several eminent mathematicians interested in problems of teaching (Dieudonné, 1973; Feller, 1957; Polya, 1981; Silva, 1964; Wilder, 1967) have argued that the development and education of intuitions should be an essential goal of mathematics teaching.

This study assumed that intuitive reasoning is used to process much of the information regarding graph interpretation and sought to assess the intuitive basis for concepts related to functional reasoning such as independent and dependent variable, linearity, and rate of change.

Theoretical Framework and Definitions

In this study functional reasoning refers to the mental processes associated with the notion of functional dependence, that is, with the notion that in some circumstances the value assumed by a certain variable depends in
Of course, it is possible to learn some mathematics long before one can function at the logical-analytical level. But to Fischbein, intuitive thinking is not merely an inferior level of operating, a "substitute form" until the mind is able to work at logical-analytical level. Instead, intuition plays a different, essential, and irreplaceable role in cognition and in productive thinking. Indeed, relying on intuition is what we do in solving most of the tasks that life presents to us at every moment. Relying essentially on intuition is also what many mathematicians do in most of their creative work (Hadamard, 1945; Lucas, 1981; Poincaré, 1913; Thom, 1973; Wilder, 1967). The intuitive level of thinking is a powerful mode of operating. As Fischbein (1982) stated:

Being a derived form of knowledge, like analytical thinking, intuition is able to organize information, to synthesize previously acquired experiences, to select efficient attitudes, to generalize verified reactions, to guess, by extrapolation, beyond the facts at hand. The greatest part of the whole process is unconscious and the product is a crystallized form of knowledge which, like perception, appears to be self-evident, internally structured and ready to guide action (p. 12).

For Fischbein the intuitive and logical-analytical functions play a complementary role. After having traced a general path, the mathematician has to check the results. The arguments must be given a logical form and the validity of the deductions must be verified. Rather than putting intuition and logic in opposition, Fischbein claimed that a superior synthetic form of mathematical understanding can
valuable auxiliary means—in many instances even a necessary support—for the functioning of operations. Images can play the role of springboards for deduction, enabling one "to outline in rough what the operations extend and bring to conclusion" (1971, p. 379). Piaget and Inhelder maintained that the collective sign system (which includes ordinary language) is not sufficient to perform all acts of thought because it conveys poorly many forms of intimate experience and is restricted to the description of concepts or singular items. This gives a particular significance to the symbolic role of images. These conclusions indicate that, besides logical-deductive thinking, some forms of intuitive thinking may also be of prime importance in mathematics.

Other writers have also discussed the role of imagery as a system of symbolic representation distinct from the discursive-verbal and have identified imagery as especially relevant in problem-solving processes and in creative thinking (Bruner & Kenney, 1965/1975; Goldin, 1983; Paivio, 1971, 1978). These distinct roles assigned to verbal and visual imagery are consistent with accounts of outstanding mathematicians and scientists concerning the processes they use in creative work (Hadamard, 1945; Higginson, 1982; Shepard, 1978) and are supported by Krutetskii's (1968/1976) investigation of mathematical abilities of gifted schoolchildren.
Skemp (1971, 1979) described conceptual structures or schemes as mental networks of concepts with two functions: to integrate existing knowledge and to serve as mental tools for the development of new knowledge. Cross-linkages of concepts may be of various types, involving relations, relations of relations, and transformations.

There are schemes for different kinds of psychological functions from sensori-motor to conceptual activities. For Skemp, to understand something is to assimilate it into an appropriate scheme. Understanding is felt subjectively and is usually not an all-or-nothing state but a matter of degree. Understanding increases with the introduction of more elements into a scheme or with an internal reorganization of a scheme, and the acquisition of understanding is conceived as a dynamic, never-ending process.

Understanding may fail to take place for several reasons: (a) the wrong scheme may be used; (b) the gap between the new idea and the (appropriate) existing scheme may be too great; and (c) the existing scheme may not be capable of assimilating new ideas without itself undergoing accommodation.

Schemes need to be developed and reformulated in order to assist the development of knowledge. However, according to Skemp, schemes also tend to behave conservatively. In situations where they are not appropriate, their tendency...
many difficulties in problems involving the interpretation of graphs.

To stimulate the creation of new conceptual structures and the harmonic development of logical, algorithmic, procedural, and intuitive thinking is one of the fundamental problems in the psychology of mathematics learning. This study focused on processes and structures related especially to intuitive thinking in the context of situations involving graphs and functional reasoning. The following sections review and discuss psychological theories related to this view of mathematical thinking processes.

Operations and Structures

Development was viewed by Piaget (1964a) as a natural and spontaneous process concerning the totality of the structures of the child's knowledge. Development is an internal process, whereas learning is induced by the influence of external situations and occurs as a function of the phases of development.

Piaget's (1964a; Piaget & Inhelder, 1971) conception of knowledge was based on the notion of operation. An operation is an internalized action that is reversible and integrated with other operations into a total structure. These operational structures form the foundations for all knowledge. Piaget distinguished four stages in the development of operational structures: sensori-motor,
3. Intuitive reasoning. The steps are usually not well defined. They come in varying orders, often superimposed. The operations are sometimes difficult to observe or distinguish—they are mostly associations, substitutions, generalizations, etc. Intuitive reasoning is the fundamental driving force in mathematical creation and learning and has been extensively discussed by Fischbein (1973, 1979, 1982).

A mixed form of thinking is procedural reasoning. It is in many respects similar to algorithmic reasoning but is more flexible and contains some aspects of intuitive reasoning. Usually the number of steps is not very large, and the concepts manipulated may be not so concrete as in algorithmic reasoning.

In mathematics teaching logical reasoning and intuitive reasoning tend to be stifled under the overwhelming emphasis given to algorithms and procedures. Students tend to develop a highly distorted view of mathematics and of how it is created and applied. Most of them think that mathematics is essentially a set of rules—to solve any mathematical problem one just needs to remember and apply the suitable rules (Carpenter, Corbitt, Kepner, Lindquist, & Reyes, 1983; Skemp, 1971). This constitutes a strictly procedural perspective on mathematics.

Visual and symbolic imagery. In developing and processing concepts and conceptual structures we use a variety of symbolic means. Some of them have an accentuated
identity, set membership, and equivalence under given conditions, or making comparisons in ordered sets. Geometric operations include projections, transformations, and assessments of congruence. Operations such as joining and taking away are related to the arithmetic operations of addition and subtraction. Other operations have essentially a logical nature. Examples are negation, conjunction, disjunction, implication, equivalence, and universal and existential quantification.

Mental operations of a different kind are substitution, association, and generalization. In substitution, a concept is replaced by an equivalent or a more restricted concept. In association, a concept is connected in some way with another concept or conceptual structure. For example, a concept may be related to a second concept that shares some common features or contrasts in some sense with the first. In generalization, a statement valid in some domain is taken to be valid in a larger domain.

Situations and problems are thought of by the human mind in terms of conceptual structures (Skemp, 1971, 1979). A conceptual structure is a network of concepts oriented towards a certain kind of response. Sometimes distinct and even conflicting conceptual structures may be used to solve a certain problem or similar problems. Some conceptual structures are weak, that is, operating with them is an insecure process that may be easily disrupted. Other
CHAPTER 4

METHOD

This study was concerned with studying the performance of different groups of students on several aspects of graph comprehension. With this purpose data was gathered using a conventional assessment instrument.

The study also sought to describe students' reasoning strategies and conceptual difficulties. The review of research suggested that spontaneous intuitive thought processes can be successfully addressed with nontraditional tasks performed by the student in individual interaction with the investigator. To gather data concerning students' thinking processes a set of graphing tasks was administered in an interview format. Students' written commentaries on test questions were also used as a source of data concerning their difficulties and strategies.

This chapter describes the sample, instruments, procedure, delimitations, and methods of data analysis used in the investigation and reports the results of reliability and validity studies.
The high school students were in classes that were made available in the schools contacted by the researcher in Athens, Georgia, and neighboring counties. In two high schools all the 12 mathematics classes with a significant enrollment of 11th graders were included in the testing sample. As this yielded a much larger group of high ability than of low ability students, additional students in 5 low ability classes were tested in two other schools. The total sample included students from 11 classes for students of above-average ability (Algebra II, Pre-calculus, and Geometry) and 6 classes for students of below-average ability (Foundations of Mathematics, Consumer Mathematics, and Elementary Geometry).

The nature of the courses in which the 11th graders in the testing sample were enrolled suggested their partition into two groups: (a) nonalgebra students, who had never enrolled in Algebra II (76 students), and (b) algebra students, who were taking or had already taken Algebra II (103 students). This partition of the sample into algebra and nonalgebra students closely parallels the traditional division of American high school students into college bound and noncollege bound students. However, since students of very low ability commonly do not take any mathematics courses in the 11th-grade, the group of nonalgebra students in the testing sample did not contain many of these students.
Interview Sample

Students who were selected for the interviews were chosen in an attempt to cover a wide range of ability, as measured by a Test of Graph Reading and Interpretation. Eighteen high school students were interviewed. Volunteers to participate in the interviews were primarily sought among the 11th graders who had taken the test. However, as there were only 11 eleventh-grade volunteers, 7 high school students from other grades in the same schools were also interviewed. In addition, 8 preservice secondary school teachers who had taken the test of graph reading and interpretation were interviewed.

Instruments

The research instruments were a test of graph reading and interpretation and a set of open-ended tasks involving the construction or interpretation of graphs.

Test of Graph Reading and Interpretation

This test, which was developed especially for the study, was used to gather data concerning levels and patterns of achievement of high school students and preservice teachers on several aspects of graph comprehension. The test was also used to contrast the students in the testing
improve the content validity, clarity, and wording of the questions.

The test contained five subtests. Four subtests corresponded to different aspects of the framework described in chapter 2 and one subtest was composed of the National Assessment items. The subtests are briefly described, following which are reported the results of the reliability and validity studies.

**Graph Reading (10 items).** This subtest corresponds essentially to the understanding and use of the basic conventions and principles involved in Cartesian graphs. Items 1, 2, and 3 are context-free questions. Items 5 and 6 deal with a discrete function. And Items 16, 18, 19, 20, and 28 deal with continuous functions.

**Graph construction (1 item).** This aspect of graph comprehension is addressed by Item 15. Graph construction was not judged to be appropriately assessed by multiple-choice questions and was mostly investigated in the interviews.

**Variation (13 items).** This subtest referred to the understanding of the representations of change and non-change processes. It especially concerned the notions of increase, decrease, constancy, and maximum. Items 4, 26, 29, and 30 are interpretation questions. Items 7, 8, 9, and 17 are application questions. And Items 10, 22, 27, 31, and 32 are multistep problems.
Table 1
Alpha Coefficients and Standard Errors of the Mean for Three Subtests

<table>
<thead>
<tr>
<th></th>
<th>Eleventh Graders</th>
<th>Preservice Teachers</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Nonalgebra</td>
<td>Algebra</td>
</tr>
<tr>
<td></td>
<td>n = 76</td>
<td>n = 103</td>
</tr>
<tr>
<td></td>
<td>Elementary</td>
<td>Secondary</td>
</tr>
<tr>
<td></td>
<td>n = 52</td>
<td>n = 31</td>
</tr>
</tbody>
</table>

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Graph Reading</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(10 items)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Alpha coefficient</td>
<td>.500</td>
<td>.467</td>
</tr>
<tr>
<td>Standard error of the mean</td>
<td>.019</td>
<td>.012</td>
</tr>
</tbody>
</table>

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Variation</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(13 items)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Alpha coefficient</td>
<td>.647</td>
<td>.651</td>
</tr>
<tr>
<td>Standard error of the mean</td>
<td>.024</td>
<td>.017</td>
</tr>
</tbody>
</table>

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Variation in Variation</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(4 items)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Alpha coefficient</td>
<td>.419</td>
<td>.454</td>
</tr>
<tr>
<td>Standard error of the mean</td>
<td>.031</td>
<td>.027</td>
</tr>
</tbody>
</table>

Note. Standard errors of the mean refer to a 0-1 scale.
Graphing Tasks

The conceptual processes and difficulties experienced by the students in problems involving functional reasoning and graph comprehension were addressed with tasks carried out in one-to-one semi-structured interviews. Some tasks used in these interviews had the starting point in a ready-made graph and others in the construction of a graph based on an imaginary situation. Several tasks contributed information regarding the same aspects of graph comprehension in a triangulation process of data collection (LeCompte & Goetz, 1982; Reichardt & Cook, 1979). The handouts used in the interviews along with sample questions are shown in Appendix B. Here is a succinct description of each task:

1. Bacteria Growth. This task was inspired by a similar situation used by Janvier (1978). It focused on graph reading and was initially designed as computer-based. Due to difficulties in access to microcomputers in some high schools it was also adapted to be presented as a ready-made graph. A sample graph generated by the computer program may be found in Appendix B. Underlying this task is a quasi-smooth relationship between two continuous variables. The students are presented with a graphical simulation of an experiment showing how the number of bacteria in a test tube varied with time and their reaction to a new kind of special food. This task was intended to study graph reading, variation, and rate of change.
made by the researcher. It was intended to study graph construction, variation, and rate of change.

5. Cost of Telephone Calls. This task focused on graph construction and interpretation. Underlying it is a step function, with time as the independent variable and cost as the dependent variable. The students were given a written statement and a table and asked to construct the corresponding graph. It was intended to study the understanding of variables, the notion of variation, the conceptual structure for linearity, and the understanding of forms of variation in an unfamiliar context.

6. Dripping Tap Experiments. This task focused on graph construction and interpretation. It was inspired by a similar task from Piaget (Piaget et al., 1968) and Swan (1980). In this task linear, piecewise linear, and nonlinear functions were to be represented. These functions were to describe the relationship between time and height of water in different beakers submitted to a constant dripping flow. The task was intended to study the conceptual structure for linearity, interpretation of variation, and interpretation of variation in variation.

Processes of change may be relatively simple and familiar to the students or very complex and containing uncommon features. The situations employed in this investigation were designed to be as simple as possible while having the potential to provoke fruitful discussions about several aspects of graph comprehension.
The first interview focused on the students' answers to the written test. The students were asked why they had selected some answers, their degree of confidence in the responses, and their interpretations of the situations presented on the test. Sometimes, additional questions referring to test situations were also posed.

In the next two interviews the students worked on the graphing tasks. The tasks and the order of presentation differed from student to student. With the stronger students the more complex tasks involving subtle features of variation in variation were addressed quite extensively. With the weaker students more time was spent on the simpler tasks and more attention was given to the fundamental aspects of graph reading and interpretation.

The students' interviews were audiotaped. Occasionally the researcher took field notes during or immediately after the interviews. For the Bacteria Growth task, disk files were also used to record data on the performance of the 10 students who took the computer version.

The tasks performed by each student interviewed, along with information on subtest performance, are summarized in Table 2. Students who are referred to or quoted in the following chapters and who do not appear in this table were preservice elementary school teachers or algebra students who made written comments on the Test of Graph Reading and Interpretation but were not interviewed.
Delimitations and Significance

This research was constrained by a number of factors that limit the scope, range, and generalizibility of its findings. This section discusses the influence of these factors on the significance of the study.

1. The 11th graders in the testing sample were not selected by random or stratified sampling methods. They were included in the sample on the basis of availability. However, to study their achievement on several aspects of graph comprehension, the 11th graders were classified into two distinct groups according to the courses that they were taking or had already taken. This classification was assumed to be meaningful, as it closely parallels the traditional division in college bound and noncollege bound students.

The groups of elementary and secondary preservice teachers, although also not selected by random or stratified sampling methods but also on the basis of availability, represented a high percentage of the populations available. For that reason they may safely be assumed to be fairly representative of their respective populations.

The size, composition, and nature of the testing sample represents what was judged as an acceptable compromise between the objectives of randomization, representation, realism, and the means at disposal of the investigator (Kish, 1975). Although, strictly speaking, generalizations
appropriate to be used with the Test of Graph Reading and Interpretation because the intention was to insure that students would profit from the interrelations between items referring to the same situational context. It was also impossible to insure that the emotional conditions of the students in the sample of this study and in the National Assessment were the same. It is not known how and how much these factors might have affected the performance of the students in the testing sample on the National Assessment items. These circumstances must present in comparing the achievement of the sample of this study and the National Assessment samples.

4. The investigation of reasoning strategies and conceptual difficulties of the students in functional reasoning and graph comprehension was based on the interview sample. The small size of this sample and the flexibility used in the conduction of the interviews precludes generalizations beyond the sample. Although it is possible that many of the findings will also apply to other groups students, they need to be completed, made precise, quantified, and even modified by further research.

These constraints, notwithstanding, the study has a considerable significance. The theoretical framework provided a basis for integrating in a meaningful way a variety of research studies. A diverse set of challenging tasks provided information about complex thinking processes that were relatively unexplored. Quantitative and
design, an analysis of variance was conducted. Significant results concerning the sources of variance were followed by tests of all pairwise contrasts of the means with control of the family error rate for each set of contrasts (Myers, 1979, pp. 292-294).

The facility levels of the National Assessment items were used to contrast the performance of the students in the testing sample with the National Assessment samples.

Qualitative data. The interviews were transcribed and the transcripts appended to the researcher's field notes. These data and students' commentaries on the written test were then subjected to a coding process (Bogdan & Biklen, 1982; Glaser and Strauss, 1967) in which the main categories followed the framework of functional reasoning and graph comprehension described in chapter 2. More specific categories were developed as the processes of data collection and coding proceeded. This procedure parallels the method of analytical induction used in ethnographic research (Goetz & LeCompte, 1981). The complete system of final coding categories is shown in Appendix C.

The process of coding the data, when it referred to alternative categories of students' processes, was also subjected to a check of reliability by the same judges. With this purpose the system of categories was explained to them and they were given random samples of transcripts of interviews to classify in the relevant alternative
CHAPTER 5

LEVELS AND PATTERNS OF ACHIEVEMENT

This study investigated the relative difficulties of graph reading, interpretation of variation, and interpretation of variation in variation, and contrasted the relative performance of high school students and preservice teachers. This chapter reports and discusses results regarding achievement on National Assessment items related to graph comprehension and on the three categories of graph comprehension considered in this study.

Performance on the National Assessment Items

For the testing sample of 11th graders in the study the correlation between the scores on the National Assessment Subtest and the other 28 items on the Test of Graph Reading and Interpretation was .63 ($p < .001$). This correlation suggests that the 28 graph comprehension items are measuring abilities similar, at least in some respects, to the abilities assessed by the National Assessment items dealing with representation of variables.
Table 1
Facilities on National Assessment Items for the NAEP Samples and the Groups in the Sample

<table>
<thead>
<tr>
<th>Item</th>
<th>NAEP National SE</th>
<th>Eleventh Graders</th>
<th></th>
<th>Preservice Teachers</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>n = 1200</td>
<td>Nonalgebra</td>
<td>Algebra</td>
<td>Combined</td>
</tr>
<tr>
<td>11</td>
<td>.34 .28</td>
<td>.12 .44</td>
<td>.28</td>
<td>.42 .65</td>
</tr>
<tr>
<td>12</td>
<td>.36 .33</td>
<td>.17 .47</td>
<td>.32</td>
<td>.48 .65</td>
</tr>
<tr>
<td>13</td>
<td>.90 .89</td>
<td>.86 .95</td>
<td>.90</td>
<td>.92 1.00</td>
</tr>
<tr>
<td>14</td>
<td>.60 .55</td>
<td>.55 .80</td>
<td>.67</td>
<td>.75 .97</td>
</tr>
<tr>
<td>33</td>
<td>.56 .47</td>
<td>.42 .50</td>
<td>.46</td>
<td>.71 .97</td>
</tr>
<tr>
<td>34</td>
<td>.20 .15</td>
<td>.01 .54</td>
<td>.28</td>
<td>.37 .84</td>
</tr>
<tr>
<td>35</td>
<td>.16 .11</td>
<td>.01 .65</td>
<td>.33</td>
<td>.13 .77</td>
</tr>
<tr>
<td>36</td>
<td>.12 .08</td>
<td>.01 .59</td>
<td>.30</td>
<td>.08 .68</td>
</tr>
</tbody>
</table>

Note: Data on the NAEP National and Southeast samples were taken from The Second Assessment of Mathematics 1977-78: Released Exercise Set (pp. 213-335) by National Assessment of Educational Progress, 1979, Denver: Educational Commission of the States.
Achievement on Graph Reading, Variation, and Variation in Variation

Descriptive statistics summarizing the performance of the four groups of students in the testing sample on the graph comprehension subtests and the graph construction item can be found in Table 4. The nonalgebra students consistently scored well below all the other groups. The preservice secondary school mathematics teachers had the highest score in interpreting variation and in graph reading. There were no large differences between the preservice elementary school teachers, the preservice secondary school teachers, and the algebra students in graph construction and in interpreting variation in variation.

To ascertain the statistical significance of these results an analysis of variance was conducted with all the subtest scores linearly transformed to a common scale, the proportion of correct items in the subtest. Graph construction, which was only represented by one item in the test, was not considered in this statistical analysis.

Table 5 summarizes the results of the analysis of variance. There was a significant interaction between group membership and subtest ($p < .001$). This interaction indicated that the pattern of relative difficulty among the subtests was not the same for all groups of students in the sample. Given this significant interaction, a follow-up
Table 5
Summary of Analysis of Variance for Achievement in Graph Comprehension

<table>
<thead>
<tr>
<th>Source of Variation</th>
<th>Degrees of Freedom</th>
<th>Sum of Squares</th>
<th>Mean Squares</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>A (Group)</td>
<td>3</td>
<td>6.67</td>
<td>2.22</td>
<td>35.81</td>
</tr>
<tr>
<td>R/A</td>
<td>258</td>
<td>16.02</td>
<td>.06</td>
<td></td>
</tr>
<tr>
<td>B (Subtest)</td>
<td>2</td>
<td>12.99</td>
<td>6.50</td>
<td>276.57</td>
</tr>
<tr>
<td>AB</td>
<td>6</td>
<td>2.89</td>
<td>.48</td>
<td>20.51</td>
</tr>
<tr>
<td>RE/A</td>
<td>516</td>
<td>12.12</td>
<td>.02</td>
<td></td>
</tr>
</tbody>
</table>

* \( p < .001 \)
Table 6
Scaled Score Means and Significance Tests for the
Group Differences in Subtest Means

<table>
<thead>
<tr>
<th>Subtests</th>
<th>Graph (10 items)</th>
<th>Variation (13 items)</th>
<th>Variation in Variation (4 items)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Groups</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Eleventh Graders</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Nonalgebra, n = 76</td>
<td>.73</td>
<td>.53</td>
<td>.39</td>
</tr>
<tr>
<td>Algebra, n = 103</td>
<td>.87</td>
<td>.77</td>
<td>.59</td>
</tr>
<tr>
<td>Preservice Teachers</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Elementary, n = 52</td>
<td>.93</td>
<td>.80</td>
<td>.51</td>
</tr>
<tr>
<td>Secondary, n = 31</td>
<td>.91</td>
<td>.90</td>
<td>.60</td>
</tr>
</tbody>
</table>

Note: Means scores are given in a 0-1 scale. Differences in means joined by an underline were not statistically significant at the level p < .008 (one-tailed t test).
Tests of statistical significance concerning pairwise comparisons among groups on each subtest were made using two-tailed t tests for unequal size groups (Myers, 1979, p. 69-70). The family error rate was again set at a level of .10 which yielded a significance level of .006 for each single contrast. The results of these tests were statistically significant for all contrasts involving nonalgebra students. In addition, the contrasts in interpretation of variation between preservice secondary school teachers and the other three groups were statistically significant.

Again, statistically significant differences corresponded to moderate or large differences among group means. These results suggest that there was indeed a marked difference in performance between the nonalgebra and the algebra students. This difference was greatest in the subtest on interpretation of variation, followed by the subtest on interpretation of variation in variation, and finally by the subtest on graph reading.

The results also indicate that the preservice elementary school teachers performed at the same level as the algebra students. They did slightly better on the variation subtest and a little worse on the graph reading and variation in variation subtests, but none of these differences was statistically significant.

The preservice secondary school teachers outperformed all the other groups. Their performance was especially high on the interpretation of variation subtest. However, in
Group contrasts suggested that there were no significant differences between the algebra 11th graders, and the preservice elementary school teachers. They also suggested that the preservice secondary school teachers were more able than the algebra 11th graders and the preservice elementary school teachers in the interpretation of variation but not in graph reading and in the interpretation of variation in variation.

The results also suggested that, for the algebra and the nonalgebra 11th graders interpretation of variation in variation was the most difficult task, followed by interpretation of variation, and finally by graph reading. They also suggested that for the preservice secondary and elementary school teachers, although interpretation of variation in variation was the most difficult task, interpretation of variation and graph reading are at about the same level of difficulty.

The results indicated further that graph reading and interpretation of variation were the only aspects of the framework in which mastery performance was achieved by some of the groups in the study. The preservice secondary teachers had a satisfactory score on both of these subtests, but one's judgment of the performance of the preservice elementary teachers and the algebra students would depend on the particular level set for mastery performance.
CHAPTER 6

THE CARTESIAN REPRESENTATION

The Cartesian system of coordinates establishes an interrelation between geometry and algebra. Points of the plane are represented by ordered pairs of numbers, and algebraic equations in two variables are given a geometric representation. This system rests on two important ideas—
the correspondence between the points of a line and the set of real numbers and the uniqueness of the parallel or orthogonal projections of every point of the plane onto any two nonintersecting lines.

Students need to develop an intuitive understanding of these ideas, which constitute the mathematical basis for the geometrical representation of functional relationships. This chapter reports and discusses observations concerning students' comprehension and use of the principles of the coordinate system and reading and constructing simple graphs. In this and subsequent chapters the main focus is on the tasks given to the interview sample (Table 2, p. 64), but occasional references are made to the performance of the testing sample.
### Table 7
Facility Values for Graph Reading Subtest

<table>
<thead>
<tr>
<th>Items</th>
<th>Eleventh Graders</th>
<th>Preservice Teachers</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Nonalgebra (n = 76)</td>
<td>Algebra (n = 103)</td>
</tr>
<tr>
<td>Discrete context-free</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>.66</td>
<td>.87</td>
</tr>
<tr>
<td>2</td>
<td>.68</td>
<td>.91</td>
</tr>
<tr>
<td>3</td>
<td>.74</td>
<td>.89</td>
</tr>
<tr>
<td>Discrete in context</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>.97</td>
<td>.99</td>
</tr>
<tr>
<td>6</td>
<td>.96</td>
<td>.99</td>
</tr>
<tr>
<td>Continuous in context</td>
<td></td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>.18</td>
<td>.38</td>
</tr>
<tr>
<td>18</td>
<td>.84</td>
<td>.91</td>
</tr>
<tr>
<td>19</td>
<td>.67</td>
<td>.91</td>
</tr>
<tr>
<td>20</td>
<td>.79</td>
<td>.87</td>
</tr>
<tr>
<td>28</td>
<td>.82</td>
<td>.95</td>
</tr>
</tbody>
</table>
### Table 8

Distribution of the Number of Correct Responses to Items 1, 2, and 3

<table>
<thead>
<tr>
<th>Number of correct items</th>
<th>Eleventh Graders</th>
<th>Preservice Teachers</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Nonalgebra (n = 76)</td>
<td>Algebra (n = 103)</td>
</tr>
<tr>
<td>None</td>
<td>9</td>
<td>5</td>
</tr>
<tr>
<td>One</td>
<td>16</td>
<td>3</td>
</tr>
<tr>
<td>Two</td>
<td>11</td>
<td>12</td>
</tr>
<tr>
<td>All</td>
<td>40</td>
<td>83</td>
</tr>
</tbody>
</table>
Table 9
Types of Difficulty in Working with Scales

<table>
<thead>
<tr>
<th>Type of Difficulty</th>
<th>High School Students</th>
<th>Preservice Secondary Teachers</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Ready-made Graphs</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Disregarding the units of measurement</td>
<td>6/9</td>
<td>2/5</td>
</tr>
<tr>
<td>Confounding the units of measurement</td>
<td>2/9</td>
<td>0/5</td>
</tr>
<tr>
<td>Incorrectly subdividing the scale units</td>
<td>7/9</td>
<td>4/8</td>
</tr>
<tr>
<td><strong>Constructed Graphs</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Indecision about the kind of graph</td>
<td>2/11</td>
<td>0/6</td>
</tr>
<tr>
<td>Nonuniformity or questionable uniformity</td>
<td>5/11</td>
<td>2/6</td>
</tr>
<tr>
<td>Hesitation or failure in the choice of the units of measurement</td>
<td>3/11</td>
<td>1/6</td>
</tr>
</tbody>
</table>

**Notes.** Data on Ready-made Graphs come from Bacteria Growth and discussion of Peter's Journey test items. Data on Constructed Graphs come from Journey to School, Parachute Jump, and Cost of Telephone Calls. In each cell is given the number of students having each difficulty against the total number of students who performed tasks or discussed test items where the difficulty might have occurred. Some students were included in more than one cell.
units. Table 9 also contains data on the occurrence of those difficulties in the interviews. The last two difficulties were especially apparent in the low ability students.

Ciro and Rote were undecided on several occasions about choosing a bar or a line graph. Ciro thought that in most situations either of them could be used interchangeably. Iori and Dema, although not addressing in an explicit way the issue of the kind of graph that they were using, represented a continuous variable by putting the units inside grid boxes instead of representing them by marks. These students were probably influenced by their previous work with histograms.

Many students used the metric ruler that they were given to draw the axes, but few of them used it to set up the scales. Some students were clearly concerned with constructing uniform scales, setting the scale marks equidistant from each other. Others made more sketchy scales whose uniformity was questionable or even difficult to recognize.

Hesitation in the choice of the units was most apparent in problems involving cost as one variable, with some students wondering if they should use 4, 5, or 10 cents or 1 dollar as the main unit. Iori provided an extreme example of lack of planning in the choice of units. She just represented all possible values on the vertical axis, starting with 1 all the way up to 43 where she ran out of paper. At
Table 10

Inaccuracies in Reading Ready-Made Graphs

<table>
<thead>
<tr>
<th>Type of Difficulty</th>
<th>High School Students</th>
<th>Preservice Secondary Teachers</th>
</tr>
</thead>
<tbody>
<tr>
<td>Estimation of the value of the function</td>
<td>6/9</td>
<td>2/8</td>
</tr>
<tr>
<td>Estimation of the value of the abscissa</td>
<td>8/11</td>
<td>2/8</td>
</tr>
</tbody>
</table>

*Note.* Data for values of the function come from Bacteria Growth, Photocopy Prices, and discussion of Rose's Experiment test items. Data for values of the abscissas come from Bacteria Growth and discussion of Boiling Water test items. In each cell is given the number of students having each difficulty against the total number of students who performed tasks or discussed the items where the difficulty might have occurred.
**Visual distractors.** Item 16 asked for the y-coordinate of one point given its x-coordinate. The intersection of two lines in the neighborhood of that point acted as a strong visual distractor. Many students had this test item wrong (Table 7). Although the intersection of those lines was rather apart from the correct point, many students were attracted by it, as they explained in the interviews:

**Canai:** "I was thinking in there, right there, that was the 1-minute point. . . . It was kind of close... And it just drove me to believe that."

**Malo:** "I was looking at the intersection of those two lines... It kind of made it like it was..."

**Miko:** "Actually looks so close..."

**Baro:** "I just assumed that it was supposed to be right on there. . . . You look at here at B and you think, well that's probably 1 minute when they intersect."

**Conclusions**

The mechanism of assigning coordinates to points was mastered by all groups of students. They were able to use it particularly well in discrete contextual situations. However, conventional features of the Cartesian system of representation sometimes seemed not to be completely present, leading to hesitation in the assignment of the x- and y-coordinates to the axes.

Careless errors in the interpretation of scales were visible in all groups of students. Conceptual difficulties in handling units of measurement and in using number line
representations of decimal numbers were reflected in a
great insecurity in dealing with scales, especially on the
part of the low ability high school students. The resist-
ance to considering points other than scale marks and mid-
points of intervals handicapped many students in making
accurate readings on the graphs.

Reading inaccuracies were quite common even among the
preservice secondary school teachers. The students tended
not to be rigorous in processing geometric information
(performing geometric operations) and were easily misled by
visual distractors.

It would seem that in graph reading the occasional
mistakes made by the most able students resulted from a
lack of readiness in their already existing and consoli-
dated conceptual structures. In the less able students the
conceptual structures relevant to reading and constructing
graphs were weak and deficient, suffering from insuffi-
ciencies in related background knowledge.
Some students felt uncomfortable with the absence of a grid behind the lines to help them figure out the coordinates:

**Semai:** "The graphs would have been easier to measure if the whole square [grid] was given."

**Dave:** "I believe if I had a grid I would definitely know... I mean, I probably would have a better score..."

Many students did not estimate the coordinates of points when these did not fall right at the scale marks but indicated the value of the nearest scale point. This was especially the case when the scale division corresponded to one measurement unit. For example, in the interviews in the Boiling Water situation (Test Items 16-24), 1 minute was almost always indicated as the instant when Group C had the problem with the flame. Another suggestive example was Malo's indication of a rather imprecise "7 or 8 cm" as the height of the plant in Rose's Experiment (Test Items 5-10) that received no water.

Besides the scale marks some students used half units. But this restricted way of using scales often provided poor estimates. For example, Tima indicated 2 1/2 minutes as the instant when Group A would have had their water boiling.

The resistance of the students to estimating the coordinates of arbitrary points may be the result of a combination of two kinds of difficulties: difficulties in performing complex geometric operations (subdividing a scale unit) and difficulties in using decimal numbers.
that point she was really confused, without any idea of how
the situation could be remedied.

Obtaining Information from Graphs

Reading coordinates from a graph constituted by a
discrete set of points was for all groups of students in
the testing sample the easiest type of task in the test.
Reading coordinates from a graph representing continuous
functions was almost as easy (Table 7). However, the stu-
dents were very inaccurate in their readings and were
easily misled by visual distractors.

Reading inaccuracies. Inaccuracies in reading ready-
made graphs were apparent both (a) in the estimation of
values of a function and (b) in the estimation of abscissas
(values of the independent variable). The students tended
not to be very rigorous in mentally projecting points from
one line to another. This lack of rigor often led to an
inaccurate reading of coordinates. Table 10 summarizes
occurrences in the interviews in which the students gave
estimates that were more than one sixth of the scale unit
apart from the correct point. Both kinds of inaccuracies
seemed to occur at about the same frequency and to depend
mostly on the units in which the variables were scaled. For
example, subdividing a unit of 10 or a unit of 100 was
easier than subdividing a unit of 1.
end of the experiment). Some students said 45 000 instead of 50 000. In the Peter’s Journey situation (Test Items 27-32) some students interpreted the first subdivision after 6:00 as either 6:10, 6:20, or 6:30, not noticing that the 1-hour period was divided into four equal parts. This kind of difficulty was especially apparent in the low ability high school students.

Disregarding or confusing the units in graphs does not seem to involve deep conceptual problems but only insufficient attention to all the information given. When offered small hints, the students were readily able to correct their answers and normally did not repeat the same mistake. A little more practice in dealing with graphs might have been effective in resolving these difficulties. But incorrect subdivision of the units, although possibly due to lack of attention, may sometimes be related to a deficient understanding of decimal and rational numbers and their representation on the number line. Hints and explanations were not always enough to promote a clear understanding, and similar mistakes were made right away, suggesting that such difficulties have a conceptual nature. They seem not likely to be overcome if not addressed in a specific way in mathematics instruction.

**Constructed graphs.** In construction tasks the difficulties were: (a) indecision about the kind of graph, (b) construction of nonuniform scales to represent uniform variables, and (c) hesitation or failure in the choice of
appropriate subdivisions of the scale units. Table 9 summarizes the frequency of occurrence of those difficulties in the interviews.

Disregarding the units reveals a lack of attention to the information contained in the labels of the axes. For example, in the graph of Bacteria Growth the students seldom (especially in the computer version) noticed that the number of bacteria was plotted in thousands, with each scale unit corresponding to 20,000. An error that was associated with disregarding the units was to indicate the number of bacteria as 50 instead of 50,000. Disregarding the units was a common mistake made by students whether or not they had difficulty in understanding the situations involved.

Confusing the units of measurement was observed in two high school students: Leza and Lure started considering days instead of hours in the Bacteria Growth task (computer version). This difficulty sometimes reveals a lack of attention to the given information, but in this case it took place when the students were still trying to understand the situation. It would appear that this difficulty occurred as the two students diverted their attention from superficial details to concentrate on the essential aspects of the situation.

Incorrect mental subdivisions of the scale units were apparent in the same task in the estimation of the midpoint between 40,000 and 60,000 (the number of bacteria at the
considerable number of students had just one or two of Items 1, 2, and 3 right (Table 8). This pattern of performance was particularly noticeable among the nonalgebra 11th graders (36% of the total group). It could result from occasional distractions or from oscillations in the thinking process, in which students sometimes assigned the x-coordinate to one axis and other times to the other.

The following sections report and discuss difficulties that the students who were interviewed had in working with scales and in obtaining information from graphs. However, since the students in the interview sample were not a representative group and the sequence and kinds of questions posed varied from student to student, the data should be interpreted as providing only rough illustrative indicators.

Working with Scales

In the interviews some students had difficulties in working with scales, either when using ready-made graphs or when asked to construct graphs themselves. These difficulties were reflected in the ways the students obtained and represented information through graphs.

Ready-made graphs. In working with ready-made graphs there were essentially three kinds of difficulties: (a) disregarding the units in which the scale was set up, (b) confusing the units, and (c) failing to consider
Understanding the Cartesian Convention

On the Test of Graph Reading and Interpretation most students were able to find the coordinates of given points and to locate points given their coordinates, as assessed by Items 1, 2, 3, 5, and 6. In general, the preservice secondary school teachers did better than the other groups, and the preservice elementary school teachers performed in between the two 11th-grade groups (Table 7).

The understanding of the process of assigning coordinates to points of the plane seems to have been mastered by all groups. However, their performance on Items 1, 2, and 3 contrasted with their performance on Items 5 and 6 suggests that specific recall of which axis was the x- or the y-axis was sometimes a problem. This was acknowledged in the interviews by some students:

Zela: "I never remember which axis is which!"
Rita: "Oh! It was a so long time ago! I can't remember which one is supposed to go first!"
Kimy: "The only thing I couldn't remember [on the test] were the (x,y) coordinates."

A few students assigned labels to the axes in a manner opposite to the usual convention. For example, Lara recalled that the first number in an ordered pair represents the x-coordinate and the second number represents the y-coordinate but assigned the x-coordinates to the vertical axis and the y-coordinates to the horizontal axis. A
graph reading and interpretation of variation in variation the preservice secondary school teachers were not much above the algebra students, and the differences between these two groups were not statistically significant.

For a mastery level of performance set at .75, the results indicate that the algebra students and the preservice elementary and secondary teachers achieved mastery in graph reading and interpretation of variation but not in interpretation of variation in variation. This would be in accordace with the fact that most of the differences between these groups were not statistically significant (p < .006). For a more stringent level of mastery performance, .90, the results indicate that only the preservice secondary teachers achieved mastery, in graph reading and interpretation of variation. But the algebra students, although slightly below the .90 cutoff level, probably should also be considered to have mastered graph reading, because their performance was not significantly different from that of the preservice secondary teachers.

Conclusions

Overall, the testing sample of 11th graders in this study, although not exactly matching the national or Southeast samples of National Assessment, did not appear to be very different from them.
school teachers, it was moderately more difficult than graph reading for algebra students, and it was much more difficult than graph reading for nonalgebra students. Interpretation of variation in variation was by far the most difficult task. Interpretation of variation in variation was much more difficult than interpretation of variation both for preservice elementary and secondary school teachers. However, the facility gap between these two subtests was not so large for algebra and nonalgebra students.

These results suggest that indeed for both nonalgebra and algebra students the relative difficulty of the aspects of graph comprehension, as measured by the items included in the Test of Graph Reading and Interpretation, is as indicated in the framework of the study. But these results also suggest that for preservice secondary and elementary school teachers graph reading and interpretation of variation are about the same level of difficulty and are much easier than interpretation of variation in variation.

**Contrasts Between Groups**

Mean scores of the different groups in the sample on the subtests of the Test of Graph Reading and Interpretation are reported in raw form in Table 4 and in the transformed scale in Table 6.
analysis was carried out to test the effects of each variable at all the levels of the other variable.

**Contrasts Between Subtests**

The mean scores for the subtests in the common scale are shown in Table 6. For each group of students in the testing sample the statistical significance of the differences of the scores on the subtests was tested with one-tailed matched-pairs $t$ tests (Lindgren, 1976, p. 353-355; Myers, 1979, p. 303) according to the Bonferroni procedure for pairwise contrasts. The error rate per family was set at $.10$ yielding a significance level of $.008$ for each single contrast. One-tailed $t$ tests were used since the framework of the study assumed that interpretation of variation in variation was more difficult than interpretation of variation and this more difficult than graph reading. The results of these tests of statistical significance are summarized in Table 6. All the contrasts were statistically significant except the contrasts on graph reading and interpretation of variation for preservice secondary and preservice elementary school teachers.

Table 6 shows that statistically significant differences corresponded to moderate (between $.10$ and $.20$) or large ($> .20$) differences between cell means. Although interpretation of variation was very close in facility to graph reading for preservice secondary and elementary
Table 4

Raw Score Means and Standard Deviations (in Parentheses) on the Subtests by Each Group

<table>
<thead>
<tr>
<th>Subtests</th>
<th>Graph Reading (10 items)</th>
<th>Variation (13 items)</th>
<th>Variation in Variation (4 items)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eleventh Graders</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Nonalgebra, n = 76</td>
<td>7.3 (1.7)</td>
<td>6.9 (2.8)</td>
<td>1.6 (1.1)</td>
</tr>
<tr>
<td>Algebra, n = 103</td>
<td>8.7 (1.2)</td>
<td>10.0 (2.3)</td>
<td>2.4 (1.1)</td>
</tr>
<tr>
<td>Preservice Teachers</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Elementary, n = 52</td>
<td>8.4 (1.5)</td>
<td>10.4 (2.0)</td>
<td>2.1 (.9)</td>
</tr>
<tr>
<td>Secondary, n = 31</td>
<td>9.1 (.9)</td>
<td>11.7 (1.4)</td>
<td>2.4 (1.0)</td>
</tr>
</tbody>
</table>
Items 14 and 34, they performed slightly above the national facility. On Items 35 and 36, they performed above the national facility. These results suggest that, overall, the testing sample of 11th graders in this study was not very different from the National Assessment national sample.

As one would expect, the preservice secondary school teachers performed higher on all items than the other groups, including the national 11th-grade sample. It should be noted, however, that an appreciable proportion of these students were misled by a superficial reading of a table (Items 11 and 12) and had some difficulty in recalling how to respond to simple analytic geometry questions (Items 34-36).

The preservice elementary school teachers scored well below the preservice secondary teachers but had about the same pattern of scores as the algebra students in the sample on most of the items not involving analytic geometry (Items 11-14). On Item 33, a story problem involving the idea of variation in which was required the identification of the role of the variables, the preservice elementary school teachers did markedly better than the algebra students. However, the preservice elementary school teachers scored well below the algebra students on the analytic geometry items.
Performance on the National Assessment items was measured in terms of facility. Facility of an item for a given group of students is the proportion of students in that group responding correctly to the item. The performances of the students in the testing sample and the performances of a national sample and a Southeast sample of 11th graders are shown in Table 3. It should be noted that in the National Assessment results the facilities of all the items for the Southeast region (where this study was conducted) were below the national facilities.

The nonalgebra students consistently performed below the national facility on all National Assessment items, but on Item 14 they scored at the Southeast facility. These results are in agreement with the fact that most nonalgebra students belong to the lower ability group.

The algebra students scored above the national facilities on all items except Item 33, a story problem involving the identification of variables. The largest differences were on Items 34-36, which concerned the geometrical representation of linear equations. The influence of recent instruction on analytic geometry, which had occurred in most Algebra II classes not much before the administration of the Test of Graph Reading and Interpretation, was probably the main cause of such differences.

The combined sample of nonalgebra and algebra students (using a simple unweighted mean) performed very close to the Southeast facility on Items 11, 12, 13, and 33. On
categories. There was agreement with the researcher's classification in 91% of the cases, which was considered as an acceptable level of reliability for the categorization process.
qualitative data were brought together to make a comprehensive assessment of students' abilities in functional reasoning and in interpreting Cartesian graphs.

Methods of Data Analysis

In this study quantitative and qualitative data were collected. These two kinds of data were expected to complement each other in providing a basis to answer the research problems (Reichardt & Cook, 1979). This section describes the methods of data analysis and the reliability studies performed to assess the accuracy of the methods of coding qualitative data.

Quantitative data. This study sought to identify and contrast levels and patterns of achievement of four groups of students on several aspects of graph comprehension. Differences among subtest mean scores and group mean scores were to be evaluated both on the grounds of their educational significance and statistical significance.

Group membership constituted a between-subjects variable with four levels (nonalgebra and algebra 11th graders, pre-service elementary and preservice secondary school teachers). Subtest scores, which were taken as the proportion of items correct in the subtest, constituted a within-subjects variable with three levels (graph reading, interpretation of variation, and interpretation of variation in variation). Based on this repeated-measures
concerning levels of achievement on the several aspects of graph comprehension to larger populations cannot be made on statistical grounds, the results obtained may be taken as illustrative of the performance on graph comprehension of a significant sample of students.

2. The test items on the Test of Graph Reading and Interpretation were not selected from a large pool of items concerning each of the aspects of graph comprehension described in the theoretical framework of this study. Notwithstanding the favourable rating of the judges, it is impossible to make a definitive claim about their representativeness, in terms of difficulty, as items designed to measure different aspects of graph comprehension.

In the design of those items there was a major concern for content validity. For this reason items were avoided that might seem artificial to the students. Questions were sought that could appear in a natural way in the context of reasonable and comprehensible situations. To prevent as much as possible the influence of the personal biases of the investigator, many of the items were taken or adapted from test questions and tasks used in previous studies. Ultimately, only the community of mathematics educators may assess the adequacy of each set of items to measure the aspects that they were intended for.

3. The National Assessment used a complex apparatus of test administration including tape recorded instructions for each item. These procedures were not considered
### Table 2

Graphing Tasks and Subtest Scores by Student

<table>
<thead>
<tr>
<th>Group/Grade</th>
<th>Subtest Scores</th>
<th>Task</th>
</tr>
</thead>
<tbody>
<tr>
<td>TD</td>
<td>DT</td>
<td>Ph</td>
</tr>
<tr>
<td>High School Students</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Iori N-11</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Malo N-12</td>
<td>7-0-6-1-2</td>
<td>X</td>
</tr>
<tr>
<td>Tima N-12</td>
<td>8-0-5-1-3</td>
<td>X</td>
</tr>
<tr>
<td>Lara N-11</td>
<td>7-0-7-1-3</td>
<td>X</td>
</tr>
<tr>
<td>Zela N-11</td>
<td>8-0-6-1-3</td>
<td>X</td>
</tr>
<tr>
<td>Suca N-11</td>
<td>9-0-6-1-3</td>
<td>X</td>
</tr>
<tr>
<td>Lure N-11</td>
<td>9-0-7-2-3</td>
<td>X</td>
</tr>
<tr>
<td>Cana N-12</td>
<td>9-1-7-1-3</td>
<td>X</td>
</tr>
<tr>
<td>Dema A-9</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ciro A-10</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Rodo A-12</td>
<td>9-0-7-0-2</td>
<td>X</td>
</tr>
<tr>
<td>Miko A-11</td>
<td>8-0-6-3-3</td>
<td>X</td>
</tr>
<tr>
<td>Rote A-11</td>
<td>9-1-9-2-2</td>
<td>X</td>
</tr>
<tr>
<td>Sara A-11</td>
<td>9-1-9-2-6</td>
<td>X</td>
</tr>
<tr>
<td>Lina A-11</td>
<td>8-0-13-1-7</td>
<td>X</td>
</tr>
<tr>
<td>Jeno A-10</td>
<td>9-0-10-3-6</td>
<td>X</td>
</tr>
<tr>
<td>Pala A-11</td>
<td>9-1-10-2-5</td>
<td>X</td>
</tr>
<tr>
<td>Sepa A-11</td>
<td>9-0-12-3-8</td>
<td>X</td>
</tr>
<tr>
<td>Preservice Secondary Teachers</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Baro 7-0-9-2-6</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>Mara 10-1-8-1-8</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>Dave 9-1-11-1-8</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>Beha 8-1-11-2-6</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>Dina 10-0-12-2-5</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>Jane 9-1-12-3-8</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>Vito 7-1-13-3-6</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>Meba 10-1-11-3-8</td>
<td>X</td>
<td>X</td>
</tr>
</tbody>
</table>

**Note:** Groups: N—Nonalgebra students, A—Algebra Students. Subtest scores are given for graph reading (10 items), graph construction (1 item), variation (13 items), variation in variation (4 items), and National Assessment. The tasks were: TD—Test Discussion, DT—Dripping Tap Experiments, Ph—Photocopy Prices, Pa—Parachute Jump, Jo—Journey to School, BC—Bacteria Growth (computer version), BR—Bacteria Growth (ready-made graph), and Te—Cost of Telephone Calls.
Procedure

This section describes the procedure followed in the administration of the Test of Graph Reading and Interpretation and in conducting the interviews.

Test administration. The test was administered during a single class period under the direct supervision of the investigator. All the students in each class took the test whether they were 11th graders or not. The purpose of the study was explained, and the students' best cooperation was sought on a voluntary basis. Then they were given the answer sheets, test booklets, and a strip of cardboard. The students were encouraged to use the strip to help in making readings from the graphs and were instructed on how to use the answer sheets. After completing the test the students were invited to write, if they so desired, a short commentary about their reactions to the test. No student needed more than a single class period to complete the test.

Interviews. With each student there were typically three interviews, each lasting for about 25 to 30 minutes. All the interviews were conducted by the researcher. For the 11th graders the interviews took place at the students' high school, generally before or after the school hours. For the preservice teachers they took place in the College of Education whenever it suited the schedules of the student and the researcher.
2. *Photocopy Prices*. This task concerned graph construction and interpretation. Underlying it is a piecewise linear relationship between two discrete variables: number of copies made and cost. The students had to complete the graph, discuss the nature of the variables, and respond to questions involving the interpretation of the represented process of variation. Whereas in all the other tasks blank white paper was used, in this task the background was a squared grid. This task was intended to study strategies for plotting points, nature of variables, and interpretation of variation.

3. *Journey to School*. This task focused on graph construction and interpretation. It was inspired by a similar task in Swan (1980). Underlying it is an arbitrary (although smooth) relationship between two continuous variables: distance and speed. The students were called upon to construct a graph from a story illustrated by a diagram. This task was intended to study the strategies of graph construction, the comprehension of variables, the notion of variation, and the understanding of different forms of variation.

4. *Parachute Jump*. This task focused on graph construction and interpretation. Underlying it is a piecewise smooth relationship between two continuous variables: time and speed. The students constructed a graph from a written statement and then contrasted their graphs with a graph constructed by the researchers.
elementary school teachers, indicating that for these students the items in this subtest carried little consistency with each other. However, the main purpose of the subtests was not to classify individual students but to contrast group performances. The uniformly low standard error of the mean of the subtests in all groups suggested that group contrasts could be made with some confidence.

The assignment of the items to each subtest was submitted to a process of validation by two judges, both graduate students in the Department of Mathematics Education at the University of Georgia. The framework of the study was explained to them, and they were asked to classify each item regarding each aspect of graph comprehension. In case of doubt, the relevance of different aspects of the framework for a correct performance on that item was analyzed in detail in discussion between the investigator and the judges. The final decisions of the judges on all items were in agreement with the researcher's original classification.

The judges also rated the difficulty level of the set of items referring to each aspect of graph comprehension on the scale easy-fair-difficult. Both judges considered the sets for graph reading and interpretation of variation in variation to be of a fair level of difficulty. Both judges also considered the set for interpretation of variation to be slightly difficult. These judgments were regarded as globally supporting the content validity and the fairness of the set of items in each subtest.
**Variation in variation (4 items).** This subtest concerned regularities in the laws of variation of one variable and especially the concept of rate of change. It involved the notions of fast and slow change, rate of change, and comparisons of rates of change. It was assessed by Items 21, 23, 24, and 25.

**National Assessment Subtest (8 items).** This subtest included items related to graph comprehension. Items 14 and 33 concerned the algebraic notion of variable. Items 11-13 concerned the interpretation of a table. And Items 34-36 focused on geometrical representations of linear equations.

**Reliability and validity.** The reliability and the validity of the subtests that were used to make group comparisons (graph reading, variation, and variation in variation) were studied. To give a measure of the internal consistency of the subtests in assessing the performance of each student the alpha coefficient of reliability (Thorndike, 1982, p. 171) of each subtest was computed for each group. To give a measure of the accuracy of the subtests in measuring the mean performance of each group the standard error of the mean (Lindgren, 1976, p. 257) on each subtest was also computed for each group. These test statistics are shown in Table 1. The generally low alpha coefficient of reliability indicated that the classification of each individual student by the subtests should be regarded very cautiously. The alpha coefficient was extremely low in the variation in variation subtest for the preservice
sample of the study with representative samples assessed by the National Assessment of Educational Progress.

The test contains 36 items, 33 of which were in a multiple-choice format. This format was chosen to allow a reasonable number of items covering several aspects of graph comprehension to be answered in a relatively short period of time (at most one class period, even for the slower students). Some of the items were suggested by problems and tasks used in previous investigations (Hart & Johnson, 1980; Janvier, 1978; Kerslake, 1977; McKenzie & Padilla, 1983). Eight of the items (including the three that were not multiple-choice) addressed mathematical concepts or skills closely related to graph comprehension and functional reasoning and were taken from the Second Mathematics Assessment. Appendix A contains a copy of the test.

The test items developed by the researcher underwent a process of successive refinement. Modifications were suggested by pilot trials with one high school class and one college class, by pilot interviews with three high school students, and by comments from four faculty members and four graduate students at the University of Georgia. The interviews with the students were used to select distractors for several items and resulted in the replacement of some items and the rewording of others. The pilot trials with the two classes suggested the replacement of items having very low point biserial correlations with the total score. Comments from faculty and graduate students helped
Preservice teachers. The testing sample also included two groups of college students: preservice elementary and preservice secondary school teachers. These students, who were just finishing their academic requirements to begin teaching, were chosen to provide an interesting contrast with high school students. Their presence in the sample was intended to indicate how well prepared they were to address in their teaching functional reasoning and graph reading and interpretation.

The preservice secondary school teachers were 31 students (mathematics education majors) taking mathematics methods courses at the University of Georgia in the academic year 1983-1984. They constituted more than 80% of the seniors in the high school and middle school mathematics education programs in that academic year. The students not included in the sample were those who did not attend class on the days in which a test of graph reading and interpretation was administered.

The preservice elementary and middle school teachers (not mathematics education majors) were 52 students taking courses in the Department of Mathematics Education of the University of Georgia in the Winter Quarter of 1984. The preservice elementary school teachers included in the testing sample constituted about two thirds of the whole population available in that quarter.
Samples

Two samples of students were included in the study: the testing sample and the interview sample. The testing sample included 262 students divided into four groups: (a) algebra 11th graders, (b) nonalgebra 11th graders, (c) preservice elementary school teachers, and (d) preservice secondary school teachers. The interview sample was composed of 26 students divided into two groups: (a) high school students, and (b) preservice secondary teachers.

Testing Sample

High school students. High school students were selected for the study because it was assumed that they should have already developed at least intuitive notions about most of the aspects of functional reasoning and graph comprehension described in the framework. The achievement of national and Southeast representative samples of 11th graders on items related to graph comprehension was reported by the Second Mathematics Assessment of (Carpenter, Corbitt, Kepner, Lindquist, & Reyes, 1981; National Assessment of Educational Progress, 1979). Eleventh graders were selected for the testing sample of this study to compare their performance with that of the National Assessment samples.
A variable may be thought of as a general representative of all the elements of a given set. Of primary importance in mathematics are the variables whose domains are the natural numbers and the real numbers. In applications of mathematics the concern is with variable quantities in which numerical values are associated with concepts from different sciences. A functional dependence between variables is established when to each single value of one of the variables, taken as the independent variable, corresponds one and only one value of the other variable, the dependent variable.

The understanding of the notion of variable quantity and the ability to identify relationships between the variables involved in a given situation is a fundamental basis for functional reasoning. This chapter reports and discusses observations concerning students' ideas and uses of the notions of variable and functional dependence.
Iori: "Vertical."

Li: "Yes, in that vertical axis, what variable is going from 1 to 60?"

Iori was obviously confused by the last question, since she felt she had already answered it.

But the most widespread concept of variable was the identification of "variable" with "measurement unit." Many students expressed themselves in ways suggesting that they held such a concept of variable. When asked to name the variables involved in a situation, they named the unit in which those variables are usually measured, like minutes, seconds, miles, feet, miles per hour, grams, ounces, etc. Table 11 reports the frequencies with which the students in the interviews identified variables by the names of measurement units, by their abstract names, or by using one kind of name for one variable and another kind for another variable.

Iori was particularly reluctant to use other names than names of measurement units for the variables. Although some questions were posed to her using the abstract names of the variables, she kept using the concrete terminology of measurement units. All the other students, however, accepted and employed the abstract designations, using both terminologies interchangeably.

Some students seemed to have been influenced by their science courses in building up their concepts of variable.
For example, Rote spoke of "manipulated variables," and Ciro referred to variables as the "factors" involved in a given situation.

Properties of Variables

Common variables may be discrete or continuous and uniform or nonuniform. Discrete variables frequently arise in counting the number of objects pertaining to a given class, whereas continuous variables are usually related to measurements. Uniform variables are invariant under affine transformations, whereas nonuniform variables are not. The most common misunderstandings concerning properties of variables involved treating continuous variables as discrete and treating discrete variables as continuous (Table 11). Also apparent in two students was a failure to appreciate the uniform nature of some variables.

The discrete or continuous nature of a variable was in most cases well identified by the students in the interview sample. But some of them treated continuous variables as if they were discrete. This confusion was apparent, for example, in the construction of the graph of the Journey to School. Some students plotted a series of dots and gave their work as finished. They did not feel a need to represent speed as varying continuously with time.

A good number of students appreciated the discrete nature of the variables involved in the Photocopy Prices
Identification of Variables

In the interviews, in the most familiar situations the variables involved were usually correctly identified. But that was not always the case in less familiar situations. For example, in the Parachute Jump, the high school students were sometimes confused about which variables to use:

Lure: "The speed you go down would be one of them. . . . [and the other] I don't know, at least I can't think of one."

Dena: "We are going to use... How fast he is falling. . . . [and] how far he is from the ground."

Sepa: "I will use speed and the meter, the distance he fell... I think!..."

The identification of the pertinent variables in uncommon or nonstandard situations constituted a hard problem for most students. Item 15 (Table 12) ranked sixth or seventh in difficulty among the 28 graph comprehension items and was perceived by many students as one of the hardest in the test:

Ramo: "[Item 15] was one of the hardest."

Kelly: "Number 15 was harder than most of the questions."

Julio: "[Item] 15 was difficult because B and C are very similar, and only by seeing the next page made me decide on B."

It seems that preexisting conceptual structures or ways of thinking about some situations constitute strong thinking frameworks. The student tries to fit each new
**Dependent and independent variables.** The classical terminology of independent and dependent variables was unknown to Pala, an 11th grader, and to Meba, a preservice secondary school teacher. However, both of them easily understood the basis of this distinction between the roles of the variables.

**Pala:** "... Independent is by itself?... The amount of water is what he puts in here. That's his constant. The different time is what he is going to find out--how much time. The time is the [dependent] variable because he gets a different..."

Other students had a clearer idea about dependent and independent variables, but this seemed to come more from other courses than from work in mathematics: "We sort of attacked it in biology last year," said Sepa. Another student, Rote, expressed himself using a mixture of terminologies from mathematics and science:

**Rote:** "The amount of water is your manipulated variable and the one you control. ... And the time it takes is the independent [dependent!] variable... And you always want your manipulated variable on the bottom."

However, in some cases, asking the students to indicate the independent and dependent variables was a source more of confusion than of clarification. For example, Mara, who on the Parachute Jump had a correct graph, interpreted the question as a message that her graph was wrong, and modified the assignment of the variables to the axes.
her reasoning shows a reversal of the usual representation of dependence.

The abstract conceptualization of functions as sets of ordered pairs and the deemphasis on the classical terminology of independent and dependent variables in the teaching of mathematics may have had effects on the difficulties that the students experienced in recognizing the different roles of the variables in concrete situations.

Conclusions

Although a few students in the interview sample expressed rather primitive concepts of variable, confounding it with unknown or with scale, most of them seemed to possess a workable notion that a variable is something that varies and can be counted or measured. Almost none of the students interviewed appeared to feel that the distinction between an abstract quantity and the unit of measurement was important.

The low ability high school students showed little appreciation for the continuous or discrete nature of the variables. An intuitive understanding was apparent in the higher ability high school students and the preservice secondary school teachers. However, some of these students did not feel the need to express unambiguously the nature of the variables in the graph, following the common practice of representing discrete variables by line graphs.
CHAPTER 8

VARIATION

The study of the laws of variation of natural phenomena was the starting point of modern science. Patterns seen in a sequence of states were made precise by quantitative measurement. The description of their interrelations in mathematical terms was made possible with the introduction of the notion of function, at whose heart was the idea of dependence between variables.

The study of variation includes the determination of intervals of increase, decrease, and constancy, and the points of maximum and minimum value. Variation is a fundamental aspect of functional reasoning and graph interpretation. This chapter reports and discusses observations regarding how students represent in a graph the variation governing given phenomena, interpret ready-made graphs, and use the idea of variation in solving simple problems.

Representing Variation

An adequate representation in a Cartesian graph of the variation present in a given situation involves two
structure for discrete variables forced the interpretation of a continuous phenomenon in terms of discrete states. The idea of continuous variation between two states was absent as well as any intuition for its representation on the graph.

Many of the lower ability high school students needed close guidance and encouragement from the investigator to succeed in constructing the graphs. But even so they showed a great insecurity about what they were doing:

Lara: "I am not going to be able to get this... I do not even know what I am doing..."

Zela: "This must be wrong!... I don't know how to do this..."

Some of the preservice secondary teachers also had difficulty in getting started in the Dripping Tap Experiments. Baro, for example, did not understand what he was expected to do, sketching the final level of beaker B on the beaker itself instead of drawing its graph. The preservice teachers seemed to be embarrassed by not being familiar with such tasks (Beha: "I have not done this since so long...").

The dot-by-dot strategy. A strategy used by many students to represent variation, even when this variation was embodied in a continuous process of change, was to start plotting a few dots representing particular states of the process. Some students stopped after plotting a few points (Table 13). Iori took a long time to understand that, in
the Journey to School, connecting the dots would give a more adequate representation of the situation. Even after Iori connected the dots, they still retained an important meaning as a key element of the graph. When Iori realized that she needed to modify the graph to match some aspects of the story, she drew a different line and then drew new dots over the new line.

Some students explicitly intended the dots that they had plotted as auxiliary points to help in drawing a continuous line. Other students, especially the preservice secondary school teachers, took the approach of starting immediately by drawing the line (Table 13).

The pervasiveness of the dot-by-dot strategy was also apparent in that some students felt uncomfortable if the information given was not numerically precise but only qualitative: "There is not given really enough information to draw the graph," said Dema in the Journey to School.

Echoes of a dot-by-dot strategy were shown by Jane, who, uncomfortable with the schematism of the Dripping Tap Experiments, drew scales on all the axes of her handout to better visualize the form of the graphs. Thinking in purely qualitative terms seemed to pose problems for many students. They apparently felt the need for a quantitative basis to support their reasoning.

**Linearity.** Many students appeared to consider linear variation as the main form of variation. Table 13 shows the
asked for the price for 2.5 minutes they would again respond by interpolating the cost of 2 and 3 minutes.

Only high ability high school students and preservice secondary school teachers were able to distinguish—and sometimes with hesitation—between situations corresponding to linear or nonlinear variation.

**Mixing up variables.** Processes involving complex variations in the laws of variation (to be more fully discussed in chapter 9) were much harder to represent than processes just dealing with simple patterns of variation. Increased complexity in the process sometimes disrupted the basic understanding of the variables that were being used.

For example, in the Parachute Jump, Zela was trying to represent an initial phase of increasing speed, followed, after the parachute was open, by a phase of decreasing speed: She had labeled the vertical axis “speed.” She drew a steep increasing line followed by a less inclined but still increasing line. Apparently she lost track of the fact that she was representing speed and started considering speed as being a measure of the inclination of the line.

In the same task, asked about how she would have represented a zero speed, Jane just drew a horizontal line, followed by a drop, a minimum, a rising segment, and again a horizontal line. She pointed at the minimum point of her sketch as the point where the speed was zero. She also confused speed with distance.
Interpreting Variation

In the usual geometrical representation of functional relationships, absence of variation is represented by horizontal lines and variation is represented by oblique lines. Positive inclinations mean increases, and negative inclinations mean decreases. Some students did not have these ideas present in the analysis of concrete situations. This section describes difficulties they had and strategies they used in the interpretation of graphs.

Global difficulties. Many students seemed not to have an intuitive feeling that movement is represented in a time/distance graph by an oblique line. In Item 4, the most difficult item on interpretation of variation (Table 14), several students admitted alternative B as a possible legitimate response:

Marco: "It was hard to determine which graph was the right one because they all seemed to contain the same information."

Valy: "It was hard to decide between B and C, because both were very similar, but it confused me."

Anna: "Question 4 was difficult for me because two answers seemed logical (B and D)."
Some students had probably seen similar problems before and had an idea of what a graph looked like ("C looked more like a graph, going sideways like that," said Zela). But, precisely because only C and D matched the "usual image" of a graph, it is striking how so many students accepted B as a possible response.

In the interpretation of graphs some students showed a complete disregard of the variation pattern. For example, in Item 17 Tima indicated (on the graph) the point corresponding to 3.3 minutes as the point where group C had their problem with the flame because "right there was where they got really bad." Mara, a preservice secondary school teacher, chose the point corresponding to .4 minutes, apparently associating the fact that the question said "shortly after the beginning of the experiment" with the fact that there was a dot there.

On Peter's Journey (Items 27-32) relating the information given in the story with the shape of the graph was perceived as particularly difficult by many students:

Katy: "The graph of Peter's Journey got kind of confusing after looking at all the other graphs."

Tacy: "I found the most difficult questions to be those of Peter's Journey. There was something about the graph that made it more difficult to read and understand."

Mena: "The graph with Peter and the bus was hard to understand. I couldn't tell the time he stayed on the bus and the time he spent walking."

Valy: "It was hard to figure out when he actually went in the arcade and how long he rode the bus."
variation is an oblique line but by making a step-by-step interpretation of the story:

Miko: "As the elevator goes up [in C] it took 5 seconds... And this one right here [B] did not take any 5 seconds when it went up. It just went up over."

Malo: "It says it takes 5 seconds to travel one floor, from one floor to the next, so it went up 5. And it says it remains there for about 10 seconds, so I went over 10, 5 over 10. And I went up 5 more and 10 more..."

Cana: "It went up in 5 seconds, it stayed there... For 10 seconds, then it went up for 5 more seconds, then stayed there 10 seconds... [whereas B] started at 5... Yeah, it went up all in the same second."

Even some students who indicated C as the correct representation in Test Item 4 had difficulty in explaining their choice. However, a good blend of intuitive and analytical thinking was shown by Rote, an 11th grader who had recently taken a science course:

Rote: "Well, C is showing the slope... And as it stops at floor 1 it is showing that from the basement to floor 1 it takes 5 seconds... and the other graph [B] shows just a straight incline, and there is no way you can do that..."

**Difficulties with the initial phase.** In the interpretation of graphs as well as in the representation of situations the initial and final phases sometimes posed additional problems. For example, in Item 4 some students said on the interviews that they had rejected choices A and B because they did not start at zero. Also, the fact that the graph of Peter's Journey (Test Items 27-32) did not start at zero was sometimes an element of additional perplexity.
"Numbers (and tables) are easier for me to understand."

In the Photocopy Prices task, in response to the question "How much would you pay for 10 copies?" most students looked at the table to figure out the answer. Either they felt more comfortable processing numerical information than processing graphical information or they assumed that more precise values could be gathered from the table. In this task the students were asked to plot the points corresponding to 22, 23, and 24 copies. Most used a numerical strategy, either multiplying each number of copies by 4 cents or figuring out the cost for 22 copies and adding 4 cents twice. Only one preservice secondary school teacher (Baro) used a geometric strategy, connecting with a straight line the points already plotted in the graph.

Some students also used numerical strategies on Item 7, an interpolation problem in a discrete context (Table 15). They read the coordinates of the points of the graph and averaged them to obtain an estimate of the required point:

Lure: "We have 120 and 160 right down here, and I saw that 140 would be halfway in between. And for 120 is 20 and for 160 is 10. So for 140 should be 15."

Jeno: "I would just have... Get... Like we get for 120 ml and the 160. To find out where their points are, 120 is 20, 160 is 10. And I get an average, I mean the number in between the two, which would be 15."
Note: "I just take the average between 160 and 120... [which is] 15... It is more like 14.7 rather than 15... [as 160 is] 9 or 10 and 120 is 20."

Geometrical strategies. In Item 7 some students also used geometric strategies that often had a strong flavor of linearity. Trying to visualize the location of the point corresponding to 140 ml, they usually disregarded all the points in the graph except the points corresponding to 120 ml and 160 ml. In Item 9, an extrapolation problem, that flavor of linearity was also present in some responses:

Luke: "If you give more than that [200 ml] it just doesn't grow. It would die. You can tell by the way it goes down."

But in other situations the students had difficulty in interpreting geometric patterns of variation. For example, none realized how the pattern of variation of the plotted points in the graph of the Photocopy Prices could be used to find the least expensive way to make 10 copies. Even when looking at the graph was suggested they did not seem to know how to use the information contained there.

Test Item 26 was another instance in which many students had difficulty in dealing with the geometric representation of variation. As they indicated in the interviews, they did not know what to do on that item:

Alexi: "I could not focus on the relationship involving the time."

Lisa: "All the questions were easy with the exception of [Item] 26. I didn't really understand it."
The students made extensive use of numerical strategies to solve problems involving the idea of variation. Most of them seemed to be much more comfortable processing numerical rather than geometrical information and often were unable to interpret patterns of variation geometrically.
CHAPTER 9

VARIATION IN VARIATION

The simplest functions to deal with are those with a constant rate of change, the linear functions. More complex are the functions in which the rate of change always varies in a uniform way, such as the quadratic functions. Still more complex are the functions that mix patterns of acceleration and retardation. At a higher level of complexity are the functions in which, at least for specific points, it does not make sense to speak of rate of change, as with the step function at the points of discontinuity. Continuity, smoothness, and regularities such as symmetries and periodicity may be also important elements to study in variation in variation.

The study of functional relationships and the interpretation of Cartesian graphs may require the ability to understand the variations in the law of variation of one variable. Particularly important is the identification, if possible, of how the rate of change varies. This chapter reports and discusses observations concerning how students deal with the notion of rate of change, identify
and slow rates of change. After having understood what they were supposed to do, all students represented correctly the graphs for these beakers.

**Dealing with rate of change at specific points.** The students consistently preferred to indicate an interval (rather than a single point) of fastest increase or decrease. To questions framed in ambiguous terms, such as "When was the decrease faster?", typical responses were:

**Sepa:** "I guess between here and here, within this hour, between the sixth and the seventh hour."

**Jeno:** "Around six and seven..."

**Jane:** "I would say between six and seven. They are decreasing more."

Jane, a preservice secondary school teacher, showed difficulty in thinking in terms of rate of change at a single point in the context of the Bacteria Growth. When asked at what point in time the bacteria were decreasing in number fastest, she replied:

**Jane:** "Well, on one specific point in time they are not dying that much, it's more on a period of time..."

The method of comparing values at the extremes of intervals can be extended through a limiting process to define the rate of change in a single point. The students interviewed could deal easily with interval comparisons but appeared to have little idea of how that extends to a general concept of rate of change.
Demai: "[The parachutist] will probably go up very fast and when he gets to about 55 [m/s] he starts rounding off..."

Dave: "[Jim's car] is going to start here at zero and... Is probably coming up, something like that, and is going coming across... And when he gets to about the third mile, he is going to slow down just a little..."

Ciro: "[When] the food run out they [the bacteria] went back down... They are still getting around the top at first, and then they slow down a little bit, when they kind of drop, and go back to that curve."

Such descriptions of graphs could result from the fact that the students probably did not know the proper technical terms for the situations and were not comfortable using a functional vocabulary based on the notions of increase and decrease.

Comparison of Rates of Change

The simplest comparisons of rates of change involve two distinct linear functions. More complex comparisons may involve one or more nonlinear functions.

The students used two distinct strategies to compare rates of change. Some used geometric strategies, observing and comparing the inclinations of lines. Others used numerical strategies, reading and comparing the coordinates of points. Some students failed in test items involving comparisons of rates of change because they had difficulty in focusing on the specific questions asked. Others confused amount and rate. Most of the students, however, when given
### Table 16
Facility Values for Variation in Variation Subtest

<table>
<thead>
<tr>
<th>Item</th>
<th>Extrapolation</th>
<th></th>
<th>Comparison of rates of change</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>23</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>.45</td>
<td>.58</td>
<td>.52</td>
<td>.48</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>21</td>
<td>.20</td>
<td>.56</td>
<td>.38</td>
<td>.68</td>
</tr>
<tr>
<td>24</td>
<td>.21</td>
<td>.30</td>
<td>.19</td>
<td>.26</td>
</tr>
<tr>
<td>25</td>
<td>.70</td>
<td>.91</td>
<td>.96</td>
<td>.97</td>
</tr>
<tr>
<td></td>
<td>High School Students</td>
<td>Preservice Secondary Teachers</td>
<td></td>
<td></td>
</tr>
<tr>
<td>----------------------</td>
<td>----------------------</td>
<td>-----------------------------</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Geometric</td>
<td>7/16</td>
<td>2/6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Arithmetic</td>
<td>3/16</td>
<td>4/6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Not classified</td>
<td>6/16</td>
<td>0/6</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Note.** Data collected from Bacteria Growth and discussion of Test Items 21 and 24. In each cell is given the number of students using each strategy against the total number of students who performed the tasks or discussed the test items where the strategy might have been used.
The high school students mostly reasoned in terms of "angle" or "slant" (Cana, Lina, Sepa) but had no idea of slope. Some preservice secondary school teachers also did not make use of that idea, and others were confused about the relation between the slope and the inclination of a line:

Dave: "I did not see that they had different slopes."

Beha: "The slope of B is less or more than the slope of A... I would say... That this has more of a slope, wouldn't it?... Let me think..."

Although sometimes preferring to use numerical strategies in concrete situations, a few students expressed correctly the relationship between slope and rate of change:

Jeno: "I was looking at the slope of the lines... And comparing B seemed to take longer."

Lure: "The one that would be the straightest would be the fastest."

Jane: "Well, the hotter the flame, the higher the slope."

Reworking rate of change questions. Most students who failed in the test the items concerning comparison of rates of change were able in the interviews, with different levels of guidance, to identify the correct answers. Sometimes the correction was made spontaneously.

Lina, for example, had Items 21 and 24 wrong. In the interview she took some time to think again about these questions--without any reference to her responses in the test--and she gave correct answers supported by correct
of a process follows the standard pattern. That is, without the problems with the flame, the temperature of Group C would increase linearly until a certain point and then approach the boiling temperature in a decelerating manner. Item 23 was of about the same difficulty for all groups in the sample (Table 16). In the interviews most students showed a general understanding of this extrapolation problem but were not very attentive in considering all its aspects.

Prelinear thinking. The extreme case of carelessness in considering the extrapolation seemed to involve just a form of rough qualitative thinking, without paying much attention to the specific features of the rate of change. Examples of this form of thinking were found both in the high school students and in the preservice secondary school teachers. It usually went along with very inaccurate reading of the coordinates of key points of the graphs:

**Dina:** "Just from looking at how A, B, and C did start off, C seems to have an head start... So obviously it would have had started boiling before B and A... And I thought it was at 3 minutes that A had the water boiling... So my answer was 2.8..."

**Mara:** "I said 1.8 [minutes]. . . . Because they started with the warmer water to begin with, so it would increase if they didn't had the problem. . . . I was thinking that they would reach the boiling point before A. . . . but I guess I was just approximating..."

Another indication of this rough, imprecise, prelinear way of thinking was the lack of consistency with which some students responded to Item 23. For example, in the
the other a horizontal line at 100 degrees. She did not notice the slowing-down process:

I: "Is that the way this line [group A] gets to the boiling point or this line [group C] gets to the boiling point?"

Lina: "[No,] it curves..."

I: "Were you paying attention to that curving?"

Lina: "No."

I: "Why not?"

Lina: "I don't know. I just never thought about it..."

It appeared that other students tended not to notice the slowing-down processes that occurred near the boiling point. However, in the Journey to Schhol, many students paid close attention to the form in which changes in velocity should be represented.

Perceiving symmetries in variation. Perceiving symmetries in variation is an indication that some of its regularities have been grasped. For example, in the Dripping Tap Experiments, Beha spontaneously recognized the internal symmetry of beakers D and E:

Beha: "This one... It goes out and comes in... Like, it's symmetrical... So it would take the same amount of time to get from here [to here] as it would to get from here [to here]..."

Other preservice secondary school teachers also recognized external symmetries. But no high school student expressed the idea that the symmetries of the beakers should
state to another state that she represented by straight lines.

**Linear and nonlinear variation.** Some students were not very attentive to aspects of the situations that implied nonlinear variation. For example, in the Dripping Tap Experiments Miko and Ciro were much more concerned with the volume of the different beakers than with their shapes. Perceiving the distinction between linear and nonlinear variation was difficult for the high school students and even caused problems for three preservice secondary school teachers (Table 18). Some students initially overlooked the distinction between these two forms of variation but were able to establish that distinction by themselves or with a minimum of guidance.

Beaker F corresponded to a piecewise linear variation. It was represented well by all preservice secondary school teachers but by only two out of five high school students. Beaker D corresponded to a complex form of curvilinear variation. Some students (Baro, Ciro) represented its graph as a straight line. But in passing to other beakers they realized the nonlinearity:

**Baro:** "[G] it's not going to be a straight line, I don't think so... Is going to be... Pretty slow... The higher the slower..."

**Ciro:** "[F...] Ah! Oh! I'm not thinking about the shapes here!..."
Dina did not draw the nonlinear middle portion in beaker I correctly. She took a good bit of time to think about G, finally indicated that it was nonlinear, and made a correct representation. Next, she did well with D, and returning to I, she had no hesitation in recognizing that its middle portion should have been a curve instead of a straight line.

It would seem that the conceptual structures of these students for dealing with complex forms of variation were very weak at first, but given their overall good cognitive ability they were able to strengthen or develop them as they were working with new tasks.

Lina behaved in a different way. She initially drew the graph of beaker G as a straight line explaining: "It raises fastest and starts slowing down." Asked if the graph was in agreement with that description, she realized that it was not, erased it, and drew an appropriate graph for beaker G. However, next she completely misunderstood beaker P. She represented the filling of this beaker by a concave arc followed by a convex arc, asserting that the graph ought to be a curve. She did not establish a conceptual relation between the shapes of the beakers and the corresponding patterns of variation.

**Accelerated and decelerated variation.** Two important classes of nonlinear variation are accelerated and decelerated variation. Most students who made a correct distinction between linear and nonlinear variation showed...
on beaker D of the Dripping Tap Experiments, preferred to do the representation first:

**Mebra:** "First I want the increase to be... Hum, let me just draw it first and see!..."

A few students showed remarkable difficulty in representing the initial portion of the graphs for the most complex beakers, G, D, and E, in which the filling speed was never constant. These may be related to the difficulties that seem to exist in general in representing the initial phase of a phenomenon. The transition from a state of repose to a state of movement seems to imply more difficulties than the transition from one form of movement to a different form of movement. Dave, working on beaker G of the Dripping Tap Experiments, illustrates this problem:

**Dave:** "I am having trouble in starting... I want to start here... But... Is going to be something like... It's not going to cross the line but it's going to sort of curve out..."

**Smooth variation.** In real-life phenomena the transition from one form of variation to another is sometimes accomplished in a single step and other times through small gradual changes. An understanding of smooth variation implies the ability to distinguish both situations and to represent them appropriately.

The representation of how an automobile varies its movement, for example, from one phase of increasing speed to a phase of constant speed, requires the understanding of
Table 19

Representation of Smooth Variation and Discontinuity

<table>
<thead>
<tr>
<th></th>
<th>High School Students</th>
<th>Preservice Secondary Teachers</th>
</tr>
</thead>
<tbody>
<tr>
<td>Smoothness</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Not represented</td>
<td>10/11</td>
<td>2/6</td>
</tr>
<tr>
<td>Represented</td>
<td>2/11</td>
<td>6/6</td>
</tr>
<tr>
<td>Discontinuity</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Not represented</td>
<td>8/8</td>
<td>3/4</td>
</tr>
<tr>
<td>Represented</td>
<td>0/8</td>
<td>1/4</td>
</tr>
</tbody>
</table>

Note. Data for smoothness come from Journey to School and Parachute Jump. Data for discontinuity come from Cost of Telephone Calls. In each cell is given the number of students having each behavior against the total number of students who performed the task where the behavior might have occurred. Three students represented smoothness on the Journey to School but not on the Parachute Jump.
Conclusions

In very simple situations the students seemed to use implicitly an intuitive idea of rate of change. But most of them were not able to use this notion in complex situations of varying rates of change. Notably, many students did not process geometrically the information contained in the inclination of lines but resorted to numerical or rough qualitative strategies.

The conceptual structures to deal with complex variation (and with variation in variation) seemed to be at a stage of very naive development in all students. They had two main tendencies in representing and comparing rates of change: (a) to confuse amount and rate and (b) to lose track of the relevant variables in the situation.

In the representation of concrete situations most high school students did not distinguish well between linear and nonlinear variation. This distinction was usually well established by the preservice secondary school teachers who, however, had significant difficulties in representing nonlinear variation.

Smoothness was usually not taken into account by the high school students. Their attention was probably focused on the most essential structural aspects of the situations. However, smooth and nonsmooth natural processes were in general well represented by the preservice secondary
school teachers. Discontinuous variation constituted a serious problem for all students.

In summary, the preservice secondary school teachers outperformed the high school students in tasks involving the ability to understand and to represent aspects of variation in variation but still had great difficulty in situations involving complex patterns of accelerated and decelerated variation and discontinuities.
the high school students and preservice secondary school teachers. That was apparent in the Cost of Telephone Calls task, in which most students who realized the form of the graph as a step function still felt the need to connect the horizontal line segments by vertical segments (Table 19). Only one student (Jane) perceived correctly the absence of meaning—in terms of the situation—of such connections. All the others drew vertical segments that they considered a legitimate part of the graph. When asked why they had drawn the vertical segments, they gave reasons such as:

Ciro: "No matter how many seconds you go into the next minute you still going to pay the extra 80 cents."

Neba: "The second he goes over 1 minute [the cost] is going to increase 80 cents."

Students' graphs and responses illustrate well a "natural horror" of discontinuity that seems to exist in their conceptual structures. As Mara said: "I always think of a graph as a continuous line." In fact, this conception has a close parallel in the history of mathematics. A similar avoidance of discontinuity was apparent, for example, in the geometric representations of sine and cosine series appearing in Fourier's 1807 memoir on heat diffusion (Grattan-Guinness, 1980).
the notion of smoothness. Smooth changes in velocity in the curves of the Journey to School task were overlooked by many high school students but were well represented by all preservice secondary school teachers (Table 19). On the Parachute Jump, however, two preservice teachers, probably overwhelmed by the complexity of this situation, failed to take smoothness into account.

Most high school students did not reveal much sensitivity for the idea of smooth variation. The widely used dot-by-dot strategy resulted in the construction of graphs as a mechanical connection of dots (sometimes done with the ruler). This strategy created favorable conditions for losing sight of such features of the situation as smooth or nonsmooth variation.

The idea that natural phenomena may be represented by smooth curves may be temporarily overgeneralized and become a kind of blocking conceptual structure. Everything turns out to be smooth, even situations where disruptions occur. For example, Ciro represented initially the effect of a sudden change in the process of variation in a distance/time problem by a smooth transition. Asked about his representation, this student was flexible enough to correct it right away.

Discontinuous variation. The idea that phenomena usually represented by continuous variables may have points of discontinuity was hard to accept and represent for most of
remarkable difficulties in representing nonlinear variations. Typically they needed two or three attempts to get what they considered an acceptable response.

Some of the descriptions that some students gave of the most complex beakers (Table 18) were heavy on geometrical terms (Dina: "Would not be a straight line... [but] sort like an S"), whereas others had a more analytical flavor (Vito: "Quick, slow, quick"). Sometimes the nature of the nonlinear variation was verbally well described but then not correctly represented in the graph. For example, in the Dripping Tap Experiments, Jane analyzed D correctly:

**Jane:** "This first quarter is going to fill up... Is going to be faster because when it gets to the middle it gets slower and slower... And [then] faster and faster."

But she was unsuccessful in attempting to sketch the graph. The curve that she drew represented only the final phase. It seems that her intuitive understanding of the situation was framed much more in verbal-analytical terms than in geometrical terms. Beha also recognized the distinction between retarded and accelerated situations and that the corresponding graphs should be different, but then her sketch showed an initial slowing down instead of an initial increase.

In some cases the attempt to give a verbal description seemed to help the student in drawing the graphical representation. But that did not happen with Meba who, working
Table 10  
Representation of Nonlinear Variation

<table>
<thead>
<tr>
<th>Behaviors</th>
<th>High School Students</th>
<th>Preservice Secondary Teachers</th>
</tr>
</thead>
<tbody>
<tr>
<td>Represents piecewise linear variation</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Beaker F</td>
<td>2/5</td>
<td>7/8</td>
</tr>
<tr>
<td>Distinguishes linear and nonlinear variation</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Beaker D</td>
<td>0/2</td>
<td>6/8</td>
</tr>
<tr>
<td>Beaker E</td>
<td>1/1</td>
<td>7/8</td>
</tr>
<tr>
<td>Beaker G</td>
<td>1/3</td>
<td>8/8</td>
</tr>
<tr>
<td>Beaker I</td>
<td>0/2</td>
<td>5/8</td>
</tr>
<tr>
<td>Explains nonlinearity verbally</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Beaker D</td>
<td>1/1</td>
<td>7/8</td>
</tr>
<tr>
<td>Beaker E</td>
<td>1/1</td>
<td>6/7</td>
</tr>
<tr>
<td>Beaker G</td>
<td>1/4</td>
<td>6/7</td>
</tr>
<tr>
<td>Represents nonlinearity</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Beaker D</td>
<td>1/2</td>
<td>5/8</td>
</tr>
<tr>
<td>Beaker E</td>
<td>0/2</td>
<td>4/7</td>
</tr>
<tr>
<td>Beaker G</td>
<td>2/5</td>
<td>7/8</td>
</tr>
<tr>
<td>Beaker I</td>
<td>0/3</td>
<td>5/8</td>
</tr>
</tbody>
</table>

Note: Data collected from Dripping Tap Experiments. Some students, having great difficulty with the simpler beakers, were not asked to work on the more difficult ones. In each cell is given the number of students having each behavior against the total number of students who performed the corresponding part of the task.
be reflected in some way in symmetries in the corresponding graphs.

Related to the comprehension of regularities in variation in variation is the understanding of the meaning of the absence of regularities. The graph of Peter’s Journey was confusing for some students, who commented that it was a "weird graph." Others said that had never seen similar shapes in graphs before. Many students had trouble in making a correct correspondence between the parts of the story and the sections of the graph. Especially difficult was interpreting what the irregular curve in the last section of the graph could mean in terms of the situation.

Distinguishing and Representing Forms of Variation

Variation may occur in a given phenomenon in several different ways. This idea seemed to be understood in an intuitive fashion by almost all the students interviewed as they represented differently different situations or different aspects of a situation. However, the students generally showed difficulties in describing the variation verbally in terms of rate of change and in representing in a graph different patterns of variation.

For students with difficulties at the level of variation, it was especially difficult to deal with variation in variation. Such was the case of Lara, for example, for whom only one basic form of variation existed: a jump from one
interview Cana maintained that either A or B would be correct. Another student, Suca, could not remember her answer on the test—which had been A—and said that it could be either B or C.

**Linear strategy.** In Test Item 23 some students seemed to make good use of the idea of linearity:

**Miko:** "I thought of the rate they were going at 1 minute and I took it as just probably, almost... .8 seconds going up at that rate... And I just said we can draw a straight line from here to here and see what... You know just a straight line..."

**Jane:** "[Putting her strip over the first part of the graph of group C]: ... What I did was kind of put my fingers like that... I don't know... And kind of estimated... Tried to see when it reached 100... That's the way I suppose I got 2.2..."

But considering the linearity and not noting the final decelerated pattern still left room for some indecision between choices A and B:

**Jane:** "I think that if I could choose again I would choose A. Because it looks better... Let me see... It kind of depends where I put my fingers right here [the initial segment of the graph of group C]."

The strategy that students used seemed to depend on the conditions under which they operated. Prelinear strategies were very common when students described how they had handled Test Item 23. But Lina, for example, when invited to use a ruler, worked in a strictly linear fashion. She drew two intersecting straight lines, one an oblique line prolonging the initial part of the graph of group C, and
explanations. Other students who were confused in Item 21 were able to make the correct comparisons when helped to focus on the aspect "from 70 to 90 degrees." It would appear that for some students just the opportunity to work the items again, perhaps this time a little more carefully, was enough to yield a correct understanding, as if they already had the conceptual basis and all that was needed was its adequate stimulation.

The comparison of the rates of change of beakers A, B, and C in the Dripping Tap Experiments seemed to be simpler to understand than Test Items 21 and 24. One aspect that might have contributed to make the Dripping Tap an easier situation was the fact that in this task all lines started from a common point (the origin).

Perceiving and Using Regularities in Variation

The students appeared to use three different strategies in problems involving variation in variation: (a) some form of prelinear thinking, (b) a dominantly linear strategy, and (c) taking into account the features of variation in variation. In addition, they were sometimes able to perceive symmetries in the laws of variation which they used in the construction of graphs.

Regularities in variation in variation can be used to make predictions. Test Item 23 assessed the understanding that, in the absence of perturbations, the law of variation
Numerical strategies were very common in finding the intervals of fastest increase or decrease of a single curvilinear function. But some students used a similar strategy even to compare the rates of change of the linear sections of the Boiling Water situation:

**Lure:** "In the first minute C goes up 20 [degrees], but so does A. And over here... A goes from 60 to 100 in about 2 minutes and C goes from 60 to 100... In about 2 minutes.... C may be a little bit faster... It goes from under 60 to 100 in less time than A goes from 60 to 100."

**Dave:** "... And them from about... Yeah, they were just taking a little longer than 1 minute, B... This is about the 90-minute [-degree!], they are on 3 [minutes]."

In Test Item 23, Meba, a preservice secondary school teacher, also used a numerical strategy. She decomposed the graph of Group C into several parts, disregarded the "inactive period" and the associated loss in temperature, and found the correct answer to the question.

**Geometrical strategies.** For some students it was difficult to know if they were using a geometrical or numerical strategy (Table 17). However, some seemed to be thinking in unequivocally geometrical terms:

**Barbi:** "I just found where it was 70 [degrees] and up to 90, they are parallel there too... I just found the relationship for the parallel lines."

**Dina:** "First I looked at A, from 70 to 90, and [then to] C, from 70 to 90, and the graphs look that they go up at the same rate..."

**Cara:** "Because the angle of the curve... Is about parallel to... Therefore they seem to get there in about the same time..."

**MaLo:** "Well, they look about the same."
the values of a function and its rate of change was illustrated by these responses to Items 21 and 24:

**Dave:** "I guess I kind of said, well since A is over here he had to be faster [from 70 to 90 degrees]."

**Jane:** "When I looked at it I thought that [the flame of] C was stronger than [the flame of] A. . . . Because it starts at a hotter temperature."

**Dina:** "I thought [C had a stronger flame at] the beginning of the experiment [because] C is starting at 40 degrees whereas the others are less than 40."

**Canan:** "[C had a stronger flame] because... C was at 60 degrees Celsius. And A was only at about 40..."

Even some understanding of the basic relationship between slope and rate of change did not always prevent this confusion:

**Jane:** "I did realize that A and C are very close to having the same slope. I saw that. But when C came in at 40 [degrees] I thought that C had the stronger initial flame, because it came on 20 degrees more..."

It would appear that Jane's conceptual structure to deal with rate of change was not strong enough to overcome the suggestion that, since during the first minute of the experiment Group C had the water at a higher temperature, they should have had the stronger flame.

**Numerical strategies.** In problems concerning comparisons of rates of change some students used numerical strategies. They read coordinates of different points from the graphs and performed arithmetical operations with them to find their answers (Table 17).
a second opportunity to think about the items were able to respond correctly to them.

Global difficulties. Item 25 asked for the comparison of rates of change in a situation where the overall differences were readily apparent, and this item was fairly easy even for the nonalgebra students (Table 16). But the other items involving comparison of rates of change (Items 21 and 24) were among the most difficult on the test. Many students related in the interviews that in Item 21 they did not focus on the specific question asked, and in their effort to understand the situation, they were looking at the whole experiment:

Suga: "I really wasn't sure how you were supposed to look at this one... It was talking about the experiment C... Something went wrong with it... [I guess] I was trying to look [understand] at the whole thing."

Malo: "Group C they dropped, I mean... Something went wrong with their flame and they dropped... I just thought... Since group A didn't have any problem..."

Cana: "Once C had the problem with the flame it decreased the temperature... And A got to boiling faster."

Confusion between value and rate. A common strategy consistently leading to wrong answers was based on the confusion between the value of a function and its rate of change. One function increasing at a slower rate than another function at a certain point or in an interval was chosen as increasing faster because it took greater values at that point or in that interval. This confusion between
Dealing with rate of change over intervals. In the Dripping Tap Experiments three out of the five high school students had problems with the representations of the graphs of the beakers when the rate of change varied at a single point (beaker F), but all five had problems when the rate of change varied continuously over an interval (beakers G, D, and E). One student, Miko, thought first that beaker G "would stay at about the same rate" but represented that by a horizontal line. Next he seemed to realize that the rate decreased but confused it with a decrease in height:

Miko: "Oh! The height is going to decrease as it is getting wider as it goes up..."

Dave, a preservice secondary school teacher, made a similar mistake on the same beaker. His graph resembled a bell-shaped curve.

Some students seemed to be thinking in terms of "constant rate of change" when they described linear variation:

Beha: "F... [its first part] is going to be constant... and [the second part] is going to be constant again..."

Jane: "[F] is going to be constant from here to here... And then from here to here is going to increase faster... [The first part of I] is going to be pretty constant, kind of thing,..."

But when they described nonlinear variation, the students often expressed themselves in terms of not the underlying process but the appearance of the graph:
regularities in variation in variation, and use them in the resolution of problems.

Concept of Rate of Change

Many of the students seemed to use implicitly some notion of rate of change although without employing that expression in their responses. They appeared to have an intuitive idea of rate of change that they used in very simple situations, as those just involving fast and slow changes. In more complex situations the students often lost track of the variables that they were working with and of the meaning of rate of change.

Contrasting fast and slow changes. In the Bacteria Growth task all students distinguished well between situations of fast and slow change, using either numerical or geometrical strategies. Perhaps because it is a comparison that may be thought of in dichotomous terms (fast/slow), it seemed more manageable than comparisons involving continuous changes in which many students had noticeable difficulties. Most students did not use a method based on the notion of rate of change to compare increases or decreases in different intervals, but used a numerical strategy comparing the values of the function at the extremes of the interval.

The main aspect involved in beakers B and C of the Dripping Tap Experiments was to distinguish between fast
Conclusions

The influence of a conceptual structure of discreteness caused problems in the form in which some students represented variation. They either thought of continuous variables in terms of discrete states or represented the variation of continuous phenomena by a set of unconnected points on a graph. Other students used a dot-by-dot strategy to represent continuous variation. This strategy, although yielding eventually to acceptable graphs, still suggested the subordination of the conceptual structures to deal with continuous variation to a conceptual structure of discreteness.

Specific difficulties were found in dealing with the initial and final phases of continuous processes as well as with intervals of constancy. Mixing up variables—often reverting to pictorial interpretations—attested to the overall weakness of the conceptual structures of most students to deal with processes of continuous variation.

For many students, who did not produce any graphs containing curvilinear sections, linear variation appeared to be the main form of continuous variation. Linearity was found to be a relatively strong conceptual structure that often tended to determine the interpretation of more complex variation. Unfamiliar forms of variation such as the step function were correctly represented by only a very restricted number of students.
### Table 15

Strategies in Interpolation

<table>
<thead>
<tr>
<th>High School Students</th>
<th>Preservice Secondary Teachers</th>
</tr>
</thead>
<tbody>
<tr>
<td>Geometric</td>
<td>6/13</td>
</tr>
<tr>
<td>Arithmetic</td>
<td>6/13</td>
</tr>
<tr>
<td>Not Classified</td>
<td>4/13</td>
</tr>
</tbody>
</table>

**Note.** Data for this table come from Photocopy Prices and the discussion of Test Item 7. In each cell is given the number of students having each strategy against the total number of students who performed the task or discussed the item where the strategy might have been used. Some students were classified in more than one cell.
Maximum, minimum, and constancy. Most students did not have any particular difficulty in indicating points of maximum or minimum value, although (as observed in chapter 6) the estimation of coordinates was often not very accurate. In graph interpretation constancy did not seem to pose major problems once the variables present were well understood.

Use of the Idea of Variation

An understanding of the most essential features of variation may be useful in handling practical problems. All students were able to use the idea of variation, employing either geometrical or numerical strategies. But some students felt uneasy about the questions involving estimation, for which they felt the lack of a procedural method of solution:

Eric: "I was confused with [Item] 7, I didn't remember studying very much that resembled this problem."

Dino: "I felt that question 7 was a difficult one for me because I was not sure how to work that one out, so I made an educated guess at it."

Numerical strategies. The students made extensive use of numerical strategies even when geometrical approaches would have been straightforward. One of them explicitly acknowledged this preference for dealing with numbers instead of graphs:
Pictorial interpretations. The influence of pictorial interpretations of the graphs was apparent in a number of the 11th graders and preservice teachers. In Item 4 these interpretations were responsible for the choice of distractor B:

Pala: "[I chose B] because it went up and then over, I guess."

Dave: "I probably did not even look, I took a semi-glance and I was not even thinking... And I was thinking more in going straight up, like in that way..."

Mixing up variables. Distance and speed seemed to be easily confused. For example, given an alternative graph representing the Parachute Jump to compare with her sketch, Sepa interpreted a horizontal line at 6 m/s representing the last part of the fall, as indicating that the parachute was "not going down [but was] stabilized." On Peter's Journey some students also appeared to confuse distance with speed. If the vertical axis was representing speed, travelling on the bus could be thought of as a mostly horizontal line:

Note: "And right here is his bus ride, because it's a straight, even,... You know, everything constant forms a straight line [and] a bus usually goes at a constant speed."

Step-by-step interpretations. Some students who chose the correct answer in Item 4 seemed to do so because of an intuitive understanding that the usual representation of
Table 14
Facility Values for Variation Subtest

<table>
<thead>
<tr>
<th>Item</th>
<th>Interpretation</th>
<th>Application</th>
<th>Multistep problems</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Eleventh Graders</td>
<td>Preservice Teachers</td>
</tr>
<tr>
<td></td>
<td>Nonalgebra</td>
<td>Algebra</td>
<td>Elementary</td>
</tr>
<tr>
<td>n = 76</td>
<td>n = 103</td>
<td>n = 52</td>
<td>n = 31</td>
</tr>
<tr>
<td>4</td>
<td>.32</td>
<td>.56</td>
<td>.63</td>
</tr>
<tr>
<td>26</td>
<td>.28</td>
<td>.44</td>
<td>.56</td>
</tr>
<tr>
<td>29</td>
<td>.57</td>
<td>.70</td>
<td>.79</td>
</tr>
<tr>
<td>30</td>
<td>.80</td>
<td>.96</td>
<td>.88</td>
</tr>
<tr>
<td>7</td>
<td>.46</td>
<td>.73</td>
<td>.81</td>
</tr>
<tr>
<td>8</td>
<td>.61</td>
<td>.85</td>
<td>.81</td>
</tr>
<tr>
<td>9</td>
<td>.75</td>
<td>.96</td>
<td>.98</td>
</tr>
<tr>
<td>17</td>
<td>.57</td>
<td>.67</td>
<td>.75</td>
</tr>
<tr>
<td>10</td>
<td>.67</td>
<td>.85</td>
<td>.94</td>
</tr>
<tr>
<td>22</td>
<td>.54</td>
<td>.80</td>
<td>.77</td>
</tr>
<tr>
<td>27</td>
<td>.33</td>
<td>.79</td>
<td>.71</td>
</tr>
<tr>
<td>31</td>
<td>.42</td>
<td>.82</td>
<td>.88</td>
</tr>
<tr>
<td>32</td>
<td>.55</td>
<td>.88</td>
<td>.85</td>
</tr>
</tbody>
</table>
Difficulties with the initial phase. Specific difficulties seemed to relate to the initial phase, and sometimes also to the final phase, of the process of variation. This was especially apparent when the process started or ended in a state of "no change." For example, in the Journey to School many students were undecided as to whether they should connect the origin to the first plotted point (usually at 1 mile and 55 miles per hour). Most of them also finished the graph with a final speed (at mile 6) of 30 miles per hour, not being sensitive to the fact that arriving at school implied stopping the car.

Difficulties with constancy. Sometimes, representing constancy seemed to involve more problems than representing increase or decrease. Miko represented by a broken line (up, down, and up again) the modification necessary in the graph of Peter's Journey (Test Items 27-32) if the bus made one single stop halfway in its trip:

Miko: "Would be a small fluctuation... Go down and up, back up again."

In the parachute task some students (Rodo, Sepa) had the notion that during the descent there would be a period with a speed of 55 m/s, but this was not included on the graph, which just showed an increase followed by a decrease. Other students, although dealing well with the free fall with a constant speed, misrepresented the last part of the open parachute fall, representing it by an
frequencies with which students, in the interviews (a) arithmetically estimated linear interpolations for a nonlinear problem, (b) connected points in a linear pattern to represent a nonlinear situation, or (c) gave an acceptable geometrical representation of a nonlinear phenomenon. These behaviors were interpreted as indicating that the students were using linearity as a global reasoning pattern, were operating linearly mainly at a geometrical level, or were able to subordinate linearity to a more general conceptual structure for complex variation.

Some students represented by straight lines situations that corresponded to nonlinear forms of variation. As Lara said: "I am just used to straight lines." In the Cost of Telephone Calls, for example, many students, after plotting the points for 1, 2 and 3 minutes, just connected them. Others did not connect the points right way but made gestures indicating that they were experiencing an internal struggle to decide the shape of the graph. It would seem that past experience and a conceptual structure of linearity were giving them difficulties in accommodating all the aspects of the new situation.

Linearity appeared to be for some students not just a geometrical conceptual structure but also a global reasoning pattern. For these students it was hard to accept that an extra half minute of a telephone call would cost 80 cents more instead of only 40 cents more. When the situation was explained they would seem to accept it, but when
### Table 11
Representing Variation and Linearity

<table>
<thead>
<tr>
<th>Strategies for representing variation</th>
<th>High School Students</th>
<th>Preservice Secondary Teachers</th>
</tr>
</thead>
<tbody>
<tr>
<td>Plots dots only</td>
<td>7/11</td>
<td>0/8</td>
</tr>
<tr>
<td>Dot-by-dot</td>
<td>7/11</td>
<td>4/8</td>
</tr>
<tr>
<td>Draws Line</td>
<td>4/11</td>
<td>5/8</td>
</tr>
</tbody>
</table>

### Linearity

| Performs arithmetical interpolations  | 3/8                  | 1/4                           |
| Draws straight line for a nonlinear situation | 6/8                  | 1/4                           |
| Represents nonlinearity               | 2/8                  | 3/4                           |

**Note.** Data for Strategies for Representing Variation come from Journey to School, Parachute Jump, and Cost of Telephone Calls. Data for Linearity come from Cost of Telephone Calls. In each cell is given the number of students having each behavior against the total number of students who performed the task where the behavior might have occurred. Some students were classified in more than one cell.
aspects: (a) an understanding of the features of the situation itself and (b) an understanding of how those features must be described in the graph. Either aspect may pose difficulties for students. The focus of this investigation was on how students represented situations rather than on how they understood them. In the interviews, a brief discussion at the beginning of each task tried to make clear the essential features of the situation involved and to get the students started in sketching the graph. But sometimes the students' general understanding of the situation did not include important details that were discussed as the task proceeded. This section describes difficulties and strategies of the students in representing variation.

Global difficulties. A few students had difficulty getting started in the construction of the graphs. Problems in identifying the variables and the functional relationships involved might have accounted for some of the difficulties. However, for some students, global aspects concerning the nature of variation were also a source of problems.

For example, Lara seemed to be thinking of the variation of speed with time as a phenomenon that could occur in a number of distinct states. In her view, in the Journey to School, the speed would be either 50, 30, or 0 miles per hour. The transition from one state to another occurred at a single point in time and was geometrically represented by a vertical line segment. It would seem that a conceptual
Variables were very easy to identify in familiar contexts and very difficult to identify in uncommon contexts. Situations involving distance and time were easily confused with situations involving speed and time.

Although many students appeared to have an intuitive idea about independent and dependent variables, some of them did not know the conventional way of assigning the variables to the axes to reflect their roles. The inability to deal in a more conscientious way with the notions of independent and dependent variable constitutes a deficiency in functional reasoning that may handicap the students in recognizing and exploring relationships between variables.
Representation of dependence. Four high school students hesitated or assigned the independent and dependent variables to the axes in an unconventional way. Sepa, one of the brightest of the high school students interviewed, indicated this as her most serious difficulty:

Sepa: "I didn't really think that graphing was really confusing or difficult. I just had difficulty deciding (figuring out) which axis to use."

All the preservice secondary school teachers followed the usual convention, but mostly in an intuitive way. For example, in the Cost of Telephone Calls, Jane had some idea about the usual representation of dependence that made her assign time to the horizontal axis and cost to the vertical axis. But she could not explain very well why she did it that way:

Jane: "I just think of time as being horizontal. . . . And price as going up and down."

Jane indicated that she was familiar with the dependent/independent terminology but did not know that usually the independent variable is assigned to the horizontal axis and the dependent variable to the vertical axis.

Explaining how she solved Item 26, Lina said she had started from the 2-kilocalories mark on the vertical axis and looked to see which graphs had the same corresponding values on the horizontal axis. Although formally correct,
Table 12
Facility Values of Graph Construction Item

<table>
<thead>
<tr>
<th>Item</th>
<th>Eleventh Graders</th>
<th>Preservice Teachers</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Nonalgebra</td>
<td>Algebra</td>
</tr>
<tr>
<td></td>
<td>( n = 76 )</td>
<td>( n = 103 )</td>
</tr>
<tr>
<td>15</td>
<td>.38</td>
<td>.64</td>
</tr>
</tbody>
</table>
Table 11. Some of them gave appropriate reasons why the plotted points should not be connected:

**Baro:** "You can only have a certain thing... And there is no way you can have 5 1/2 copies or so..."

**Lisa:** "After all, we were looking for the number of copies and the corresponding prices."

However, in this task some students also expressed the ambiguous idea that the dots should be connected or not according to what "is easier to read."

Some students consistently took little care in constructing uniform scales, as already mentioned in chapter 6. But more serious conceptual problems with the uniformity of a variable were shown by Lara, who constructed on grid paper a scale in which nonconsecutive integers followed each other without any consideration for relative differences. Conceptual problems were also shown by Miko in the Parachute Jump. He reasoned well that "the speed falls when the parachute opens," but then got confused saying that "but the time would fall too..." And in the Journey to School instead of regarding distance as a continuously flowing quantity, this same student divided it into separate parts, each part corresponding to one section of the diagram. Then he plotted distance/speed pairs, each time referring the partial distances to the origin of the graph.
Table II

Names and Classifications of Variables

| Type of name                                      | High School Students | Preservice Secondary Teachers |
|---------------------------------------------------|----------------------|______________________________|
| Measurement units for both variables              | 4/13                 | 0/6                         |
| Mixed names                                       | 5/13                 | 5/6                         |
| Abstract names for both variables                 | 4/13                 | .                            |

Classification of Variables

| Type        | High School Students | Preservice Secondary Teachers |
|-------------|----------------------|______________________________|
| Discrete    |                      |                               |
| Correct     | 3/5                  | 2/3                          |
| Ambiguous   | 1/5                  | 1/3                          |
| Incorrect   | 1/5                  | 0/3                          |
| Continuous  |                      |                               |
| Correct     | 7/16                 | 4/4                          |
| Incorrect   | 9/16                 | 0/4                          |

Note: Data for Type of Name come from Journey to School, Parachute Jump, and Cost of Telephone Calls. Data for Classification of Discrete Variables come from Photocopy. Data for Classification of Continuous Variables come from Cost of Telephone Calls and Dripping Tap Experiments. In each cell is given the number of students having each behavior against the total number of students who performed the task where the behavior might have occurred.
The Notion of Variable

In the interviews the question "What are the variables involved in this situation?" was asked at the beginning of the construction of each graph. Responding to this question most students were able to identify—with or without help from the investigator—the variables relevant to the description of the situation. But there were two exceptions to this general pattern. One was Malo, who did not know the meaning of term "variable." The other was Miko, who, on the Parachute Jump, did not understand the question at all. He first indicated $x$ as the variable, although in the task no $x$ was referred to. Realizing that the variable had to be some quantity and not just a literal symbol, he then indicated 55 m/s and 6 m/s as the variables. It would seem that Miko was thinking of variables in terms of "unknowns," that is, the single most important pieces of information one looks for in solving a mathematical problem.

Other students, although able to identify variables, seemed to possess only a rudimentary concept of what a variable was. A very primitive idea of variable was held by Iori, who identified it with "scale," as illustrated by the following dialogue:

I: "What are the variables that you are going to use on this graph?"

Iori: "The 1 through 60 up here."

I: "Yes, that is what variable?"